Solutions to Parametric HW #1

1. Since $\frac{dy}{dt}=\frac{d}{dt}\left[t^2+6t+5\right]=2t+6$ and $\frac{dx}{dt}=\frac{d}{dt}\left[t^2\right]=2t$,

then $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{2t+6}{2t}=1+\frac{3}{t}$.

To find $\frac{d^2y}{dx^2}$, we must differentiate $\frac{dy}{dx}$ with respect to t and divide by $\frac{dx}{dt}$:

$$\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}=\frac{\frac{d}{dt}\left[1+\frac{3}{t}\right]}{\frac{dx}{dt}}=\frac{-\frac{3}{t^2}}{\frac{dx}{dt}}=\frac{-\frac{3}{t^2}}{2t}=-\frac{3}{2t^3}$$

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2. $\frac{dy}{dt}=\frac{d}{dt}\left[2t^3-t^2\right]=6t^2-2t$ and $\frac{dx}{dt}=\frac{d}{dt}\left[t^2+1\right]=2t$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{6t^2-2t}{2t}=3t-1$$

$$\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}=\frac{\frac{d}{dt}\left[3t-1\right]}{\frac{dx}{dt}}=\frac{3}{\frac{dx}{dt}}=\frac{3}{2t}$$

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3. $\frac{dy}{dt}=\frac{d}{dt}\left[3t^2+2t\right]=6t+2$ and $\frac{dx}{dt}=\frac{d}{dt}\left[\sqrt{t}\right]=\frac{1}{2}t^{-\frac{1}{2}}$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{6t+2}{\frac{1}{2}t^{-\frac{1}{2}}}=12t^{\frac{3}{2}}+4t^{\frac{1}{2}}$$

$$\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}=\frac{\frac{d}{dt}\left[12t^{\frac{3}{2}}+4t^{\frac{1}{2}}\right]}{\frac{dx}{dt}}=\frac{18t^{\frac{1}{2}}+2t^{-\frac{1}{2}}}{\frac{dx}{dt}}=\frac{18t^{\frac{1}{2}}+2t^{-\frac{1}{2}}}{\frac{1}{2}t^{-\frac{1}{2}}}=36t+4$$

4. $\frac{dy}{dt}=\frac{d}{dt}\left[t^2+t\right]=2t+1$ and $\frac{dx}{dt}=\frac{d}{dt}\left[\ln t\right]=\frac{1}{t}$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{2t+1}{\frac{1}{t}}=2t^2+t$$

$$\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}=\frac{\frac{d}{dt}\left[2t^2+t\right]}{\frac{dx}{dt}}=\frac{4t+1}{\frac{dx}{dt}}=\frac{4t+1}{\frac{1}{t}}=4t^2+t$$

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5. $\frac{dy}{dt}=\frac{d}{dt}\left[4\cos t-1\right]=-4\sin t$ and $\frac{dx}{dt}=\frac{d}{dt}\left[3\sin t+2\right]=3\cos t$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{-4\sin t}{3\cos t}=-\frac{4}{3}\tan t$$

$$\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}=\frac{\frac{d}{dt}\left[-\frac{4}{3}\tan t\right]}{\frac{dx}{dt}}=\frac{-\frac{4}{3}\sec^2 t}{\frac{dx}{dt}}=\frac{-\frac{4}{3}\sec^2 t}{3\cos t}=-\frac{4}{9}\sec^3 t$$

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6. (a) $\frac{dy}{dx}=\frac{3t^2-2t}{2t+1}$.

(b) When $t=2$, $\frac{dy}{dx}=\frac{3\cdot 2^2-2\cdot 2}{2\cdot 2+1}=\frac{8}{5}$, $x=5$ and $y=4$, so the tangent line equation is $y-4=\frac{8}{5}(x-5)$.

7. (a) $\frac{dy}{dx}=\frac{3\cos t}{-2\sin t}=-\frac{3}{2}\cot t$.

(b) When $t=\frac{\pi}{4}$, $\frac{dy}{dx}=-\frac{3}{2}\cot\frac{\pi}{4}=-\frac{3}{2}$, $x=\sqrt{2}$ and $y=\frac{3\sqrt{2}}{2}$, so the tangent line equation is $y-\frac{3\sqrt{2}}{2}=-\frac{3}{2}\left(x-\sqrt{2}\right)$.

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8. (a) $\frac{dy}{dx} = \frac{2t-4}{1}$ with $\frac{dy}{dt} = 2t-4$ and $\frac{dx}{dt} = 1$.

(b) A horizontal tangent occurs when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$, so a horizontal tangent occurs when $2t - 4 = 0$, or at $t = 2$. When $t = 2$, $x = 7$ and $y = -4$, so a horizontal tangent occurs at the point $(7, -4)$. A vertical tangent occurs when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$.

Since $1 \neq 0$, there is no point of vertical tangency on this curve.

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9. (a) $\frac{dy}{dx} = \frac{3t^2-3}{2t-1}$ with $\frac{dy}{dt} = 3t^2-3$ and $\frac{dx}{dt} = 2t-1$.

(b) A horizontal tangent occurs when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$, so a horizontal tangent occurs when $3t^2 - 3 = 0$, or when $t = \pm 1$. When $t = 1$, $x = 1$ and $y = -2$, and when $t = -1$, $x = 3$ and $y = 2$, so horizontal tangents occur at the points $(1, -2)$ and $(3, 2)$.

A vertical tangent occurs when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$ so a vertical tangent occurs when $2t-1=0$, or $t = \frac{1}{2}$. When $t = \frac{1}{2}$, $x = \frac{3}{4}$ and $y = -\frac{11}{8}$, so a vertical tangent occurs at the point $\left(\frac{3}{4}, -\frac{11}{8}\right)$.

10. (a) $\frac{dy}{dx} = \frac{4\cos t}{-2\sin t} = -2\cot t$ with $\frac{dy}{dt} = 4\cos t$ and $\frac{dx}{dt} = -2\sin t$.

(b) A horizontal tangent occurs when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$, so a horizontal tangent occurs when $4\cos t = 0$, or at $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$. When $t = \frac{\pi}{2}$, $x = 3$ and $y = 3$, and when $t = \frac{3\pi}{2}$, $x = 3$ and $y = -5$, so horizontal tangents occur at the points $(3, 3)$ and $(3, -5)$.

A vertical tangent occurs when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$, so a vertical tangent occurs when $-2\sin t = 0$, or when $t = 0$ and $t = \pi$. When $t = 0$, $x = 5$ and $y = -1$, and when $t = \pi$, $x = 1$ and $y = -1$, so vertical tangents occur at the points $(5, -1)$ and $(1, -1)$.
