# AP Calculus BC Saturday Study Session #2: Particle Motion

**(With special thanks to Lin McMullin & Wes Gordon)**

Particle motion and similar problems are on the AP Calculus exams almost every year. The particle may be a "particle," a person, a car, or some other moving object. The position, velocity or acceleration may be given as an equation, a graph or a table and sometimes you will be given an initial condition to work with. You may be asked about the motion of the particle: its direction, when it changes direction, its maximum position in one direction, etc. Speed, the absolute value of velocity, is also a common topic.

**What you should know how to do:**

- Move easily between the position, velocity and acceleration equations by differentiating or integrating.
- If you are given the velocity and an initial position, or given the acceleration and an initial velocity, you may often be able to approach the problem as an accumulation problems using the following equations:

$$x(t_1) = x(t_0) + \int_{t_0}^{t_1} v(t)\,dt$$

$$v(t_1) = v(t_0) + \int_{t_0}^{t_1} a(t)\,dt$$

- Speed is the **absolute value** of velocity (it is not a vector quantity). In other words, speed $= |v(t)|$.
  - If the velocity and acceleration have the same sign, the speed is increasing.
  - If the velocity and acceleration have different signs, the speed is decreasing.
- Graphically, speed is the non-directed distance from the velocity graph to the $t$-axis. If the distance of the velocity is increasing then speed is increasing. Reflecting the parts of the velocity graph that lie below the $t$-axis will give you the graph of the speed.
- The total distance traveled at velocity $v(t)$ from $t = a$ to $t = b$ is given by $\int_a^b |v(t)|dt$.
- The net distance (a.k.a, displacement) over the same interval is $\int_a^b v(t)dt$.
- Don't be reluctant to use your graphing calculator for these problems if you're permitted to do so. Integrating an absolute value by hand can be tricky but your calculator can do it with ease!

# Parametric Particle Motion (BC Only)

Particle motion problems on the AP Calculus BC exam are often in the context of parametric equations or in the context of vectors.

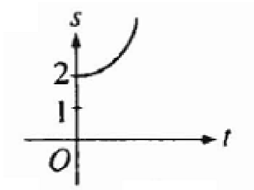
Suppose that a particle has a position vector given by $(x(t), y(t))$ at time $t$.

- **Velocity:** $v(t) = (x(t), y'(t)) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$
- **Speed (a.k.a, magnitude of velocity):** $|v(t)| = \sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$
- **Acceleration:** $a(t) = (x''(t), y''(t)) = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}\right)$
- **Distance Traveled between $\boldsymbol{t = a}$ and $\boldsymbol{t = b}$:** $\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$
- **Parametric Definition of Slope:** $\dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$
- **Parametric Interpretations of Particle Motion:**
  - $\frac{dx}{dt} < 0 \iff$ The particle is moving left
  - $\frac{dx}{dt} > 0 \iff$ The particle is moving right
  - $\frac{dy}{dt} < 0 \iff$ The particle is moving down
  - $\frac{dy}{dt} > 0 \iff$ The particle is moving up
  - $\frac{dx}{dt} = 0 \iff$ The particle's position graph has a vertical tangent
  - $\frac{dy}{dt} = 0 \iff$ The particle's position graph has a horizontal tangent
  - $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0 \iff$ The particle is not moving

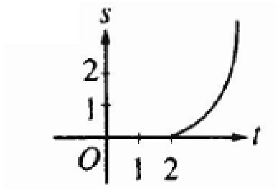
# Multiple Choice Questions

1. **1998 #90 (BC) - Calc OK:** A particle starts from rest at the point (2, 0) and moves along the $x$-axis with a constant positive acceleration for time $t \geq 0$. Which of the following could be the graph of the distance $s(t)$ of the particle from the origin as a function of time $t$ ?

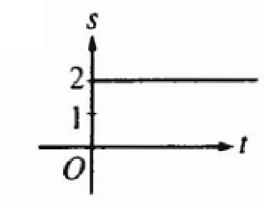
a.




c.




e.

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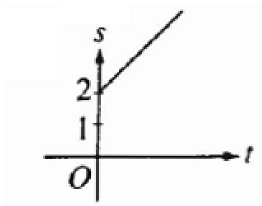


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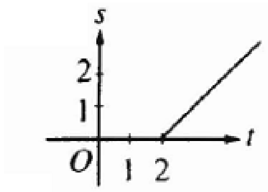
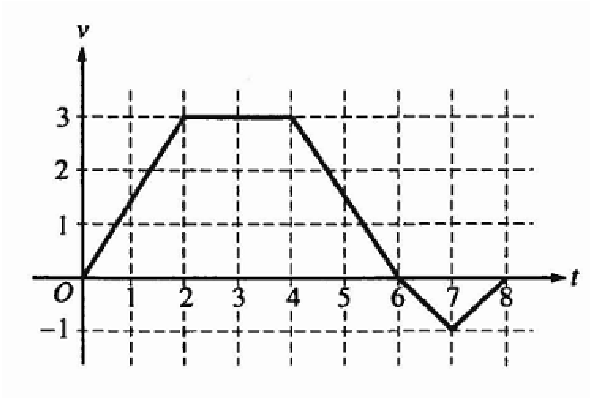



2. **1997 #8 (AB but suitable for BC) - No Calc:**




A bug begins to crawl up a vertical wire at time $t = 0$. The velocity $v$ of the bug at time $t$, $0 \leq t \leq 8$, is given by the function whose graph is shown above. At what value of $t$ does the bug change direction?

a. 2 b. 4 c. 6 d. 7 e. 8

3. **1997 #9 (AB but suitable for BC) - No Calc:** Using the scenario and graph from the previous question, what is the total distance the bug traveled from $t = 0$ to $t = 8$?

a. 14 b. 13 c. 11 d. 8 e. 6

4. **2003 #83 (AB but suitable for BC) - Calc OK:** The velocity, in ft/sec, of a particle moving along the $x$-axis is given by the function $v(t) = e^t + te^t$. What is the average velocity of the particle from time $t = 0$ to time $t = 3$?

a. 20.086 ft/sec
b. 26.447 ft/seo
c. 32.809 ft/sec
d. 40.671 ft/sec
e. 79.342 ft/sec

5. **1997 #79 (BC) - Calc OK:** The position of an object attached to a spring is given by $y(t) = \frac{1}{6}\cos(5t) - \frac{1}{4}\sin(5t)$, where $t$ is time in seconds. In the first 4 seconds, how many times is the velocity of the object equal to 0?

a. Zero
b. Three
c. Five
d. Six
e. Seven

6. **2003 #25 (AB but suitable for BC) - No Calc:** A particle moves along the $x$-axis so that at time $t \geq 0$ its position is given by $x(t) = 2t^3 - 21t^2 + 72t - 53$. At what time $t$ is the particle at rest?

a. $t = 1$ only
b. $t = 3$ only
c. $t = \frac{7}{2}$ only
d. $t = 3$ and $t = \frac{7}{2}$
e. $t = 3$ and $t = 4$

7. **1998 #14 (AB but suitable for BC) - No Calc:** A particle moves along the $x$-axis so that its position at time $t$ is given by $x(t) = t^2 - 6t + 5$. For what value of $t$ is the velocity of the particle zero?

a. 1
b. 2
c. 3
d. 4
e. 5

8. **2003 #91 (BC) - Calc OK:** The height $h$, in meters, of an object at time $t$ is given by $h(t) = 24t + 24t^{\frac{3}{2}} - 16t^2$. What is the height of the object at the instant when it reaches its maximum upward velocity?

a. 2.545 meters
b. 10.263 meters
c. 34.125 meters
d. 54.889 meters
e. 89.005 meters

9. **2003 #76 (AB but suitable for BC) - Calc OK:** A particle moves along the $x$-axis so that at any time $t \geq 0$, its velocity is given by $v(t) = 3 + 4.1\cos(0.9t)$. What is the acceleration of the particle at time $t = 4$?

a. -2.016 b. -0.677 c. 1.633 d. 1.814 e. 2.978

10. **1998 #24 (AB but suitable for BC) - No Calc:** The maximum acceleration attained on the interval $0 \leq t \leq 3$ by the particle whose velocity is given by $v(t) = t^3 - 3t^2 + 12t + 4$ is

a. 9 b. 12 c. 14 d. 21 e. 40

11. **2003 #87 (BC) - Calc OK:** A particle moves along the $x$-axis so that at any time $t \geq 0$, its velocity is given by $v(t) = \cos\left(2 - t^2\right)$. The position of the particle is 3 at time $t = 0$. What is the position of the particle when its velocity is first equal to 0?

a. 0.411 b. 1.310 c. 2.816 d. 3,091 e. 3.411

12. **2003 #91 (AB but suitable for BC) - Calc OK:** A particle moves along the $x$-axis so that at any time $t > 0$, its acceleration is given by $a(t) = \ln\left(1 + 2^t\right)$. If the velocity of the particle is 2 at time $t = 1$, then the velocity of the particle at time $t = 2$ is

a. 0.462 b. 1.690 c. 2.555 d. 2.886 e. 3.346

13. **1997 #87 (AB but suitable for BC) - Calc OK:** At time $t \geq 0$, the acceleration of a particle moving on the $x$-axis is $a(t) = t + \sin t$. At $t = 0$, the velocity of the particle is $-2$. For what value of $t$ will the velocity of the particle be zero?

a. 1.02 b. 1.48 c. 1.85 d. 2.81 e. 3.14

14. **1997 #13 (BC) - No Calc:** A particle moves along the $x$-axis so that its acceleration at any time $t$ is $a(t) = 2t - 7$. If the initial velocity of the particle is 6, at what time $t$ during the interval $0 \leq t \leq 4$ is the particle farthest to the right?

a. 0 b. 1 c. 2 d. 3 e. 4

# PARAMETRIC MOTION

15. **2003 #4 (BC) - No Calc:** For $0 \le t \le 13$, an object travels along an elliptical path given by the parametric equations $x = 3\cos t$ and $y = 4\sin t$. At the point where $t = 13$, the object leaves the path and travels along the line tangent to the path at that point. What is the slope of the line on which the object travels?

a. $\frac{-4}{3}$ b. $\frac{-3}{4}$ c. $\frac{-4\tan 13}{3}$ d. $\frac{-4}{3\tan 13}$ e. $\frac{-3}{4\tan 13}$

16. **2003 #7 (BC) - No Calc:** The position of a particle moving in the $xy$-plane is given by the parametric equations $x = t^3 - 3t^2$ and $y = 2t^3 - 3t^2 - 12t$. For what values of $t$ is the particle at rest?

a. $-1$ only c. 2 only e. $-1$, 0, and 2

b. 0 only d. $-1$ and 2 only

17. **2003 #84 (BC) - Calc OK:** A particle moves in the $xy$-plane so that its position at any time $t$ is given by $x(t) = t^2$ and $y(t) = \sin(4t)$. What is the speed of the particle when $t = 3$?

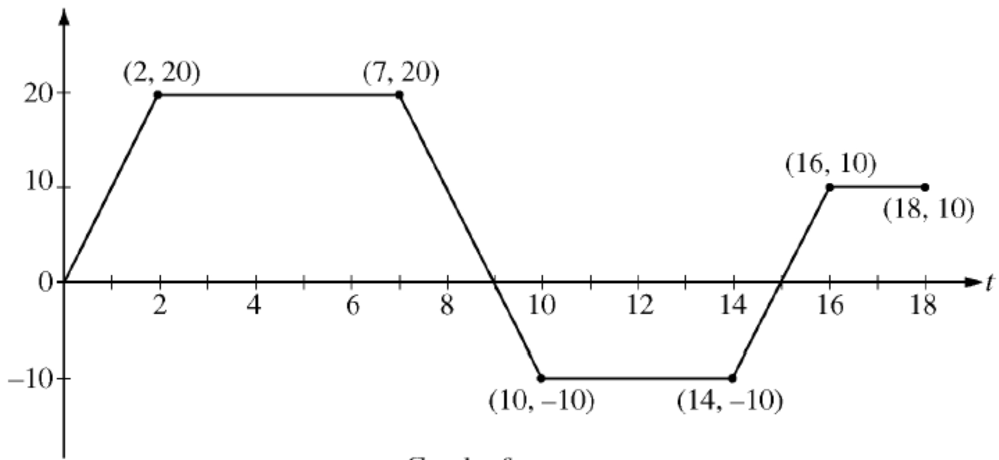
a. 2.909 b. 3.062 c. 6.884 d. 9.016 e. 47.393

18. **1998 #10 (BC) - No Calc:** A particle moves on a plane curve so that at any time $t > 0$, its $x$-coordinate is $t^3 - t$ and its $y$-coordinate is $(2t-1)^3$. The acceleration vector of the particle at $t = 1$ is

a. (0, 1) b. (2, 3) c. (2,6) d. (6, 12) e. (6, 24)

# Free Response Questions

## ❶ 2010B #4 (BC) – No Calc




Graph of $v$

A squirrel starts at building $A$ at time $t = 0$ and travels along a straight, horizontal wire connected to building $B$. For $0 \le t \le 18$, the squirrel's velocity is modeled by the piecewise-linear function defined by the graph above.

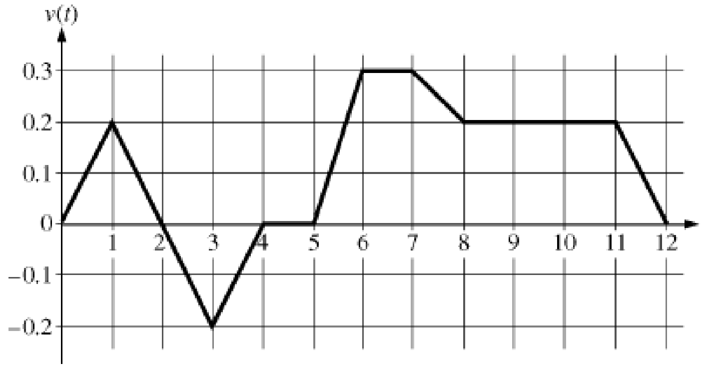
(a) At what times in the interval $0 < t < 18$, if any, does the squirrel change direction? Give a reason for your answer.

(b) At what time in the interval $0 \le t \le 18$ is the squirrel farthest from building $A$? How far from building $A$ is the squirrel at that time?

(c) Find the total distance the squirrel travels during the time interval $0 \le t \le 18$.

(d) Write expressions for the squirrel's acceleration $a(t)$, velocity $v(t)$, and distance $x(t)$ from building $A$ that are valid for the time interval $7 < t < 10$.

## ❷ 2009 #1 (BC) – Calc OK




Caren rides her bicycle along a straight road from home to school, starting at home at time $t = 0$ minutes and arriving at school at time $t = 12$ minutes. During the time interval $0 \le t \le 12$ minutes, her velocity $v(t)$, in miles per minute, is modeled by the piecewise-linear function whose graph is shown above.

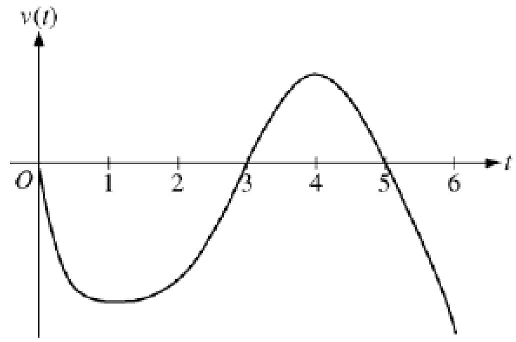
(a) Find the acceleration of Caren's bicycle at time $t = 7.5$ minutes. Indicate units of measure.

(b) Using correct units, explain the meaning of $\int_0^{12} |v(t)|\, dt$ in terms of Caren's trip. Find the value of $\int_0^{12} |v(t)|\, dt$.

(c) Shortly after leaving home, Caren realizes she left her calculus homework at home, and she returns to get it. At what time does she turn around to go back home? Give a reason for your answer.

(d) Larry also rides his bicycle along a straight road from home to school in 12 minutes. His velocity is modeled by the function $w$ given by $w(t) = \frac{\pi}{15}\sin\left(\frac{\pi}{12}t\right)$, where $w(t)$ is in miles per minute for $0 \le t \le 12$ minutes. Who lives closer to school: Caren or Larry? Show the work that leads to your answer.

## ❸ 2008 #4 (BC) – No Calc




Graph of $v$

A particle moves along the $x$-axis so that its velocity at time $t$, for $0 \le t \le 6$, is given by a differentiable function $v$ whose graph is shown above. The velocity is 0 at $t = 0$, $t = 3$, and $t = 5$, and the graph has horizontal tangents at $t = 1$ and $t = 4$. The areas of the regions bounded by the $t$-axis and the graph of $v$ on the intervals $[0, 3]$, $[3, 5]$, and $[5, 6]$ are 8, 3, and 2, respectively. At time $t = 0$, the particle is at $x = -2$.

(a) For $0 \le t \le 6$, find both the time and the position of the particle when the particle is farthest to the left. Justify your answer.

(b) For how many values of $t$, where $0 \le t \le 6$, is the particle at $x = -8$ ? Explain your reasoning.

(c) On the interval $2 < t < 3$, is the speed of the particle increasing or decreasing? Give a reason for your answer.

(d) During what time intervals, if any, is the acceleration of the particle negative? Justify your answer.

## ❹ 2012 #2 (BC) – Calc OK

For $t \geq 0$, a particle is moving along a curve so that its position at time $t$ is $(x(t), y(t))$. At time $t = 2$, the particle is at position $(1, 5)$. It is known that $\frac{dx}{dt} = \frac{\sqrt{t+2}}{e^t}$ and $\frac{dy}{dt} = \sin^2 t$.

(a) Is the horizontal movement of the particle to the left or to the right at time $t = 2$ ? Explain your answer. Find the slope of the path of the particle at time $t = 2$.

(b) Find the $x$-coordinate of the particle's position at time $t = 4$.

(c) Find the speed of the particle at time $t = 4$. Find the acceleration vector of the particle at time $t = 4$.

(d) Find the distance traveled by the particle from time $t = 2$ to $t = 4$.

## ❺ 2010 #3 (BC) – Calc OK

A particle is moving along a curve so that its position at time $t$ is $(x(t), y(t))$, where $x(t) = t^2 - 4t + 8$ and $y(t)$ is not explicitly given. Both $x$ and $y$ are measured in meters, and $t$ is measured in seconds. It is known that $\frac{dy}{dt} = te^{t-3} - 1$.

(a) Find the speed of the particle at time $t = 3$ seconds.

(b) Find the total distance traveled by the particle for $0 \le t \le 4$ seconds.

(c) Find the time $t$, $0 \le t \le 4$, when the line tangent to the path of the particle is horizontal. Is the direction of motion of the particle toward the left or toward the right at that time? Give a reason for your answer.

(d) There is a point with $x$-coordinate 5 through which the particle passes twice. Find each of the following.

(i) The two values of $t$ when that occurs

(ii) The slopes of the lines tangent to the particle's path at that point

(iii) The $y$-coordinate of that point, given $y(2) = 3 + \frac{1}{e}$

## ❻ 2010B #2 (BC) – Calc OK

The velocity vector of a particle moving in the plane has components given by

$$\frac{dx}{dt} = 14\cos\left(t^2\right)\sin\left(e^t\right) \text{ and } \frac{dy}{dt} = 1 + 2\sin\left(t^2\right), \text{ for } 0 \le t \le 1.5.$$

At time $t = 0$, the position of the particle is $(-2, 3)$.

(a) For $0 < t < 1.5$, find all values of $t$ at which the line tangent to the path of the particle is vertical.

(b) Write an equation for the line tangent to the path of the particle at $t = 1$.

(c) Find the speed of the particle at $t = 1$.

(d) Find the acceleration vector of the particle at $t = 1$.