NATIONAL MATH + SCIENCE INITIATIVE

# Euler's Method
# (BC Only)

Euler's Method is used to generate numerical approximations for solutions to differential equations that are not separable by methods tested on the AP Exam. It is necessary to know an initial point and a rate of change (the derivative) of the function. Euler's Method uses locally linear approximations at successive steps to estimate a solution.

Required information: $(x_0, y_0)$ the initial point

$\frac{dy}{dx}$ the derivative of the function; usually given implicitly.

$dx$ the differential increment for $x$.

Starting at the initial point $(x_0, y_0)$, the next point can be found using the formulas

$$x_1 = x_0 + \Delta x \text{ and } y_1 = y_0 + \Delta y$$

But since we don't know the exact value for $\Delta y$ , we can approximate it by using $dy$ when the increment $\Delta x$ is small. This gives us

$$x_1 = x_0 + dx \text{ and } y_1 = y_0 + f'(x_0, y_0) * dx.$$

The process is repeated as many times as necessary to find the final solution using the general formulas

$$x_{n+1} = x_n + dx \text{ and } y_{n+1} = y_n + f'(x_n, y_n) * dx$$

Although you may have had to do four or five iterations on a homework problem, the AP Exam has traditionally asked for only two iterations. This allows you to demonstrate that you understand the process while not having to get lost in the algebra of repeating the process many times. An easy way to keep track of all the computations is to list the values in a simple table like the one shown below.

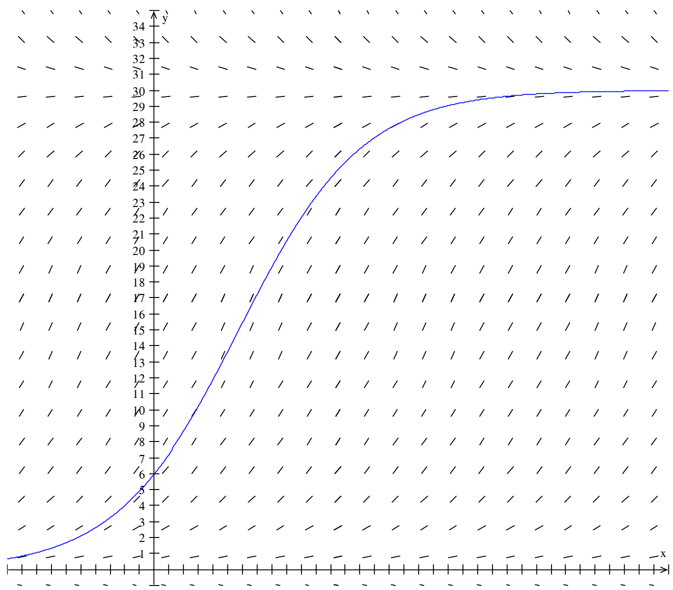
| Original $x$ | Original $y$ | $\Delta x = dx$ | $\Delta y \approx dy = f'(x, y) * dx$ | New $x$ = Original $x$+ $dx$ | New $y$ = Original $y$ + $dy$ |
|---|---|---|---|---|---|
| | | | | | |
| | | | | | |

You must be given the value of $dx$ or be given an interval and the number of increments that will allow you to compute the value of $dx$.

# Logistic Growth Functions
## (BC Only)

Logistic growth functions look like exponential functions at first which follow the differential equation $\frac{dP}{dt} = kP$. However, if there is a maximum carrying capacity for the local environment, we introduce a limiting factor $(M - P)$. This modifies the differential equation to $\frac{dP}{dt} = kP(M - P)$.

A slope field for a logistic curve looks like the following



And the solution follows an *S*-shaped graph. The solution of the general logistic differential equation given by

$$\frac{dP}{dt} = kP(M - P) \quad \text{is} \quad P = \frac{M}{1 + ae^{-(Mk)t}}$$

Where $M$ is the maximum carrying capacity and $k$ is the growth constant ($M$ and $k$ are both positive). Once again, the algebra to solve this type of problem is very messy, takes a lot of time, and is very difficult to grade accurately. So, previous AP exams have *not* asked students to find a solution. Instead the exams ask a few questions that test a student's understanding of the logistic growth concepts. The two questions most frequently asked are

1) to find the limit of the population over a long period of time. This is always the maximum carrying capacity, regardless of the initial population.

2) to determine when the population is growing the fastest. This is always when the population is half of the maximum carrying capacity. Note that this is an output value, not an input value of time.

Topic: Euler's Method and Logistic Growth Functions
Student Saturday Session

## I. Euler's Method – Multiple Choice Examples

Example 1 2003 BC5

5. Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = x + y$ with the initial condition $f(1) = 2$. What is the approximation for $f(2)$ if Euler's method is used, starting at $x = 1$ with a step size of 0.5 ?

(A) 3 (B) 5 (C) 6 (D) 10 (E) 12

Example 2 2008 BC7

7. Given that $y(1) = -3$ and $\frac{dy}{dx} = 2x + y$, what is the approximation for $y(2)$ if Euler's method is used with a step size of 0.5, starting at $x = 1$ ?

(A) $-5$ (B) $-4.25$ (C) $-4$ (D) $-3.75$ (E) $-3.5$

## II. Euler's Method – Free Response Examples

Example 1 2001 BC5 part b

Let $f$ be the function satisfying $f'(x) = -3f(x)$, for all real numbers $x$ with $f(1) = 4$ .

a) Use Euler's Method, starting at $x = 1$, with step size of 0.5, to approximate $f(2)$.

Example 2 2007 Form B BC5 parts (c) and (d)

Consider the differntial equation $\frac{dy}{dx} = 3x + 2y + 1$

c) Let $y = f(x)$ be a particular solution to the differential equation with intial condition $f(0) = -2$.

Use Euler's Method, starting at $x = 0$ with step size of $\frac{1}{2}$, to approximate *f(1)*.

d) Let $y = g(x)$ be another solution to the differential equation with initial condition $g(0) = k$, where $k$ is a constant. Euler's Method, starting at $x = 0$ with step size of 1, gives the approximation $g(1) = 0$. Find the value of $k$.

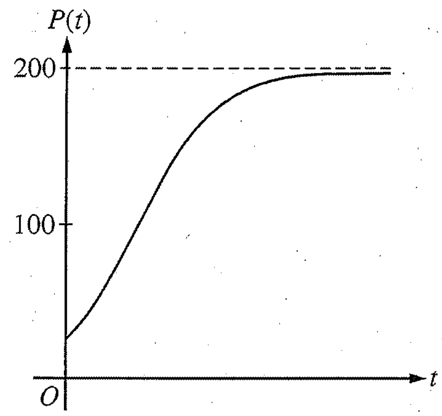
III. Logistic Growth Functions – Multiple Choice

Example 1 2003 BC21

21. The number of moose in a national park is modeled by the function $M$ that satisfies the logistic differential equation $\frac{dM}{dt} = 0.6M\left(1 - \frac{M}{200}\right)$, where $t$ is the time in years and $M(0) = 50$. What is $\lim_{t\to\infty} M(t)$ ?

(A) 50 (B) 200 (C) 500 (D) 1000 (E) 2000

Example 2 2008 BC 24




24. Which of the following differential equations for a population $P$ could model the logistic growth shown in the figure above?

(A) $\frac{dP}{dt} = 0.2P - 0.001P^2$

(B) $\frac{dP}{dt} = 0.1P - 0.001P^2$

(C) $\frac{dP}{dt} = 0.2P^2 - 0.001P$

(D) $\frac{dP}{dt} = 0.1P^2 - 0.001P$

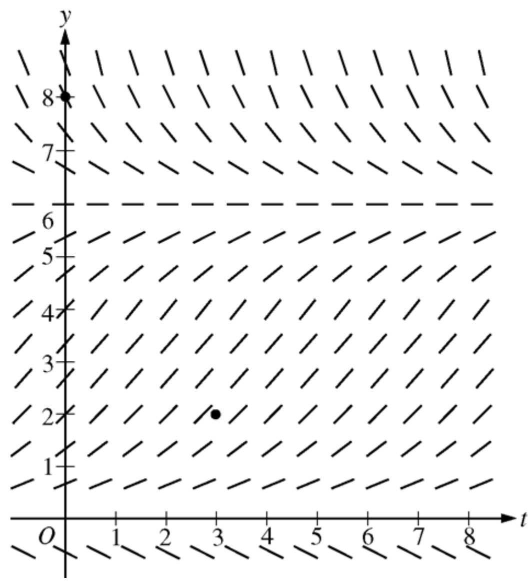
(E) $\frac{dP}{dt} = 0.1P^2 + 0.001P$

IV. Logistic Growth Functions -- Free Response Examples

1. 2008 BC6 part (a)

Consider the logistic differential equation $\frac{dy}{dt} = \frac{y}{8}(6 - y)$. Let $y = f(t)$ be the particular solution to the differential equation with $f(0) = 8$.

a) A slope field for this differential equation is given below. Sketch the possible solution curves through the points (3, 2) and (0, 8).



Example 2 2004 BC5 parts (a) and (b)

A population is modeled by a function $P$ that satisfies the logistic differential equation

$$\frac{dP}{dt} = \frac{P}{5}\left(1 - \frac{P}{12}\right).$$

a) If $P(0) = 3$, what is $\lim_{t \to \infty} P(t)$?

If $P(0) = 20$, what is $\lim_{t \to \infty} P(t)$?

b) If $P(0) = 3$, for what value of $P$ is the population growing the fastest?