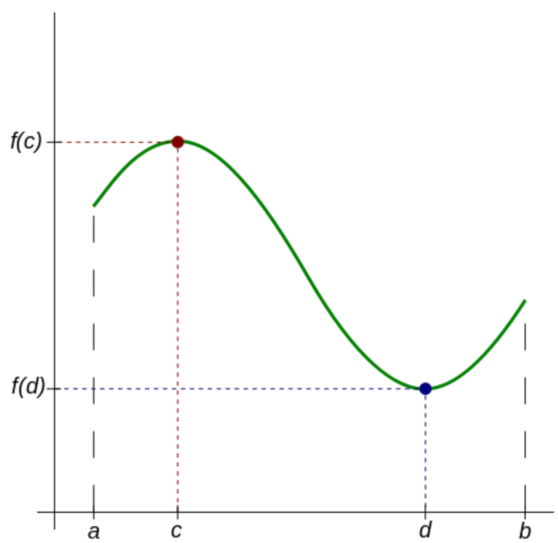
AP Calculus BC Saturday Study Session #1: The "Big" Theorems (EVT, IVT, MVT, FTC)

(With special thanks to Lin McMullin)

On the AP Calculus Exams, students should be able to apply the following "Big" theorems though students need not know the proof of these theorems.

The Extreme Value Theorem (EVT)

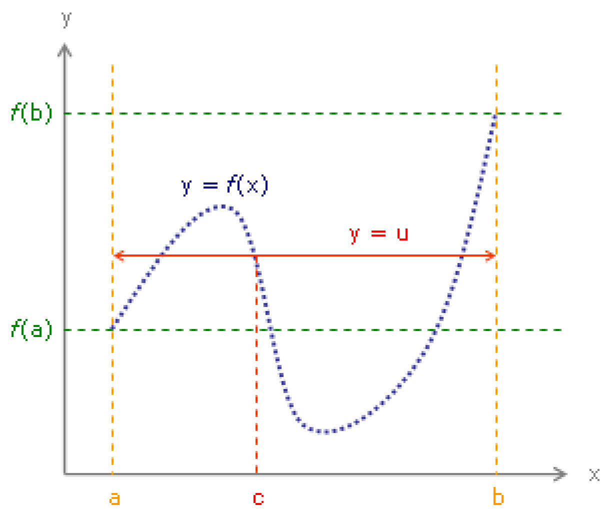
- ***Formal Statement:*** If a function f is continuous on a closed interval $[a, b]$, then:
  1. There exists a number c in $[a, b]$ such that $f(x) \leq f(c)$ for all x in $[a, b]$.
  2. There exists a number d in $[a, b]$ such that $f(x) \geq f(d)$ for all x in $[a, b]$.
- ***Translation:*** If a function f is continuous on a closed interval $[a, b]$, then f takes on a maximum and a minimum value on that interval.
- ***Picture:***



- ***Special Notes:***
  - A function may attain its maximum and minimum value more than once. For example, the maximum value of $y = \sin(x)$ is 1 and it reaches this value many, many times.
  - The extreme values often occur at the endpoint of the domain. That's why it's so important to check the endpoints of an interval when doing a maximization/minimization problem!
  - For a constant function, the maximum and minimum values are equal (in fact, all the values are equal).

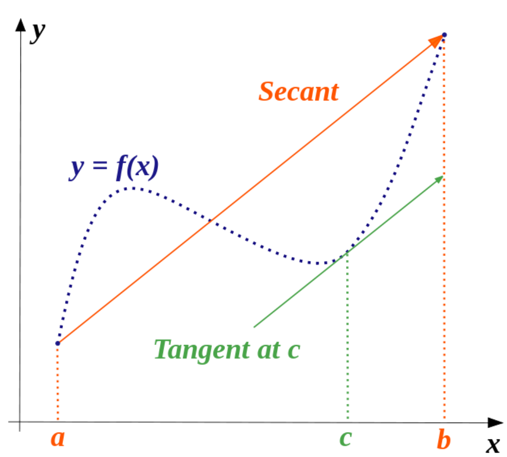
The Intermediate Value Theorem (IVT)

- ***Formal Statement:*** If a function f is continuous on a closed interval $[a, b]$ and $f(a) \neq f(b)$, then for every value of u between $f(a)$ and $f(b)$, there exist at least one value of c in the open interval (a, b) so that $f(c) = u$.
- ***Translation:*** A continuous function takes on all the values between any two of its values.
- ***Picture:***



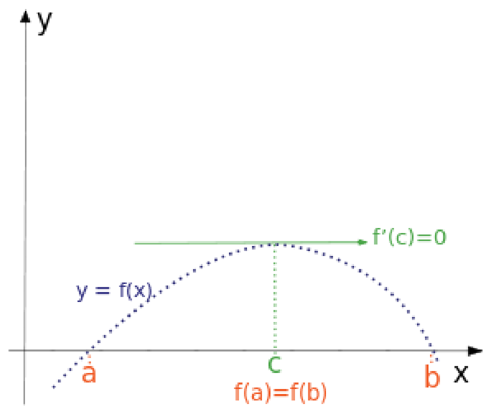
Mean Value Theorem (MVT)

- ***Formal Statement:*** If a function f is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in the open interval (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.
- ***Translation:*** If a function is continuous and differentiable, somewhere in the interval the tangent line must be parallel to the secant line between the endpoints. In other words, the instantaneous rate of change is equal to the average rate of change.
- ***Picture:***



Rolle's Theorem (a special case of the MVT)

- ***Formal Statement:*** If a function f is continuous on a closed interval $[a, b]$, differentiable on the open interval (a, b), and $f(a) = f(b)$, then there exists a number c in the open interval (a, b) such that $f'(c) = 0$.
- ***Translation:*** If a function is continuous and differentiable, the function must have a place with a horizontal tangent if there are two places where the function takes on the same value. In other words, there must be a relative minimum or maximum between two places where the function takes on the same value.
- ***Picture:***



The Fundamental Theorem of Calculus (FTC)

Assume that $f(x)$, $g(x)$, and $h(x)$ are differentiable functions and that $F(x)$ is an antiderivative of $f(x)$. In other words, $F'(x) = f(x)$.

- ***The First Fundamental Theorem of Calculus (1st FTC)***
  - $\int_a^b f(x)dx = F(b) - F(a)$.
    - **Equivalently:** $\int_a^b f'(x)dx = f(b) - f(a)$ **(sometimes this is called the NET CHANGE)**
    - **Equivalently:** $\int_a^x f'(t)dt = f(x) - f(a)$
  - This yields the incredibly useful formula $f(x) = f(a) + \int_a^x f'(t)dt$. This is used in MANY free response questions!
- ***The Second Fundamental Theorem of Calculus (2nd FTC)***
  - $\frac{d}{dx}\int_a^x f(t)dt = f(x)$
  - **Chain Rule Variation:** $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = f(h(x)) \cdot h'(x) - f(g(x)) \cdot g'(x)$

Multiple Choice Questions

EVT & IVT

1. **1997 #81 (BC) - Calc OK:** Let f be a continuous function on the closed interval [-3,6]. If $f(-3) = -1$ and $f(6) = 3$, then the Intermediate Value Theorem guarantees that

   a. $f(0) = 0$

   b. $f'(c) = \frac{4}{9}$ for at least one c between -3 and 6.

   c. $-1 \le f(x) \le 3$ for all x between -3 and 6.

   d. $f(c) = 1$ for at least one c between -3 and 6.

   e. $f(c) = 0$ for at least one c between -1 and 3.

2. **1998 #91 (AB but suitable for BC) - Calc OK:** Let f be a function that is differentiable on the open interval (1, 10). If $f(2) = -5$, $f(5) = 5$, and $f(9) = -5$, which of the following must be true?

   I. f has at least 2 zeros.
   II. The graph of f has at least one horizontal tangent.
   III. For some c, $2 < c < 5$, $f(c) = 3$.

   a. None
   b. I only
   c. I and II only
   d. I and III only
   e. I, II and III

3. **1998 #26 (AB but suitable for BC) - No Calc:**

x	0	1	2
$f(x)$	1	k	2

The function f is continuous on the closed interval [0, 2] and has values that are given in the table above. The equation $f(x) = \frac{1}{2}$ must have at least two solutions in the interval [0, 2] if $k =$

   a. 0
   b. $\frac{1}{2}$
   c. 1
   d. 2
   e. 3

MVT

4. **2003 #83 (BC) - Calc OK:**

x	0	1	2	3	4
$f(x)$	2	3	4	3	2

The function f is continuous and differentiable on the closed interval [0,4]. The table above gives selected values of f on this interval. Which of the following statements must be true?

a. The minimum value of f on [0, 4] is 2.

b. The maximum value of f on [0, 4] is 4.

c. $f(x) > 0$ for $0 < x < 4$

d. $f'(x) < 0$ for $2 < x < 4$

e. There exists c, with $0 < c < 4$, for which $f'(c) = 0$.

5. **2003 #80 (AB but suitable for BC) - Calc OK:** The function f is continuous for $-2 \le x \le 1$ and differentiable for $-2 < x < 1$. If $f(-2) = -5$ and $f(1) = 4$, which of the following statements could be false?

a. There exists c, where $-2 < c < 1$, such that $f(c) = 0$.

b. There exists c, where $-2 < c < 1$, such that $f'(c) = 0$.

c. There exists c, where $-2 < c < 1$, such that $f(c) = 3$.

d. There exists c, where $-2 < c < 1$, such that $f'(c) = 3$.

e. There exists c, where $-2 \le c \le 1$ such that $f(c) \ge f(x)$ for all x on the closed interval $-2 \le x \le 1$.

6. **1998 #4 (AB but suitable for BC) - No Calc:** If f is continuous for $a \le x \le b$ and differentiable for $a < x < b$, which of the following could be false?

a. $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ for some c such that $a < c < b$.

b. $f'(c) = 0$ for some c such that $a < c < b$.

c. f has a minimum value on $a \le x \le b$.

d. f has a maximum value on $a \le x \le b$.

e. $\int_a^b f(x)dx$ exists.

7. **2003 #92 (BC) - Calc OK:** Let f be the function defined by $f(x) = x + \ln x$. What is the value of c for which the instantaneous rate of change of f at $x = c$ is the same as the average rate of change of f over [1,4]?

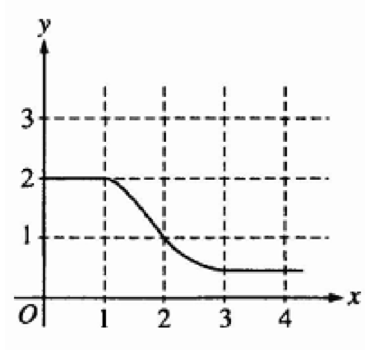
a. 0.456
b. 1.244
c. 2.164
d. 2.342
e. 2.452

FTC

8. **2003 #27 (BC) - No Calc:** $\dfrac{d}{dx}\left(\int_0^{x^3} \ln\left(t^2+1\right)dt\right) =$

a. $\dfrac{2x^3}{x^6+1}$
b. $\dfrac{3x^2}{x^6+1}$
c. $\ln\left(x^6+1\right)$
d. $2x^3\ln\left(x^6+1\right)$
e. $3x^2\ln\left(x^6+1\right)$

9. **1997 #78 (AB but suitable for BC) - Calc OK:**



The graph of f is shown in the figure above. If $\int_1^3 f(x)dx = 2.3$ and $F'(x) = f(x)$, then $F(3) - F(0) =$

a. 0.3

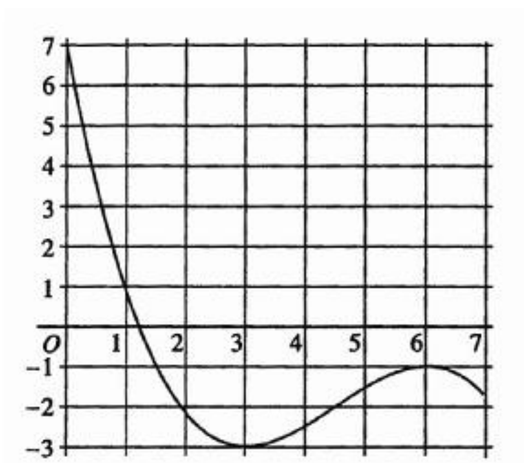
b. 1.3

c. 3.3

d. 4.3

e. 5.3

10. **2003 #18 (BC) - No Calc:**



Graph of f

The graph of the function f shown in the figure above has horizontal tangents at $x = 3$ and $x = 6$. If $g(x) = \int_0^{2x} f(t)dt$, what is the value of $g'(3)$?

a. 0

b. −1

c. −2

d. −3

e. −6

11. **1998 #28 (BC) - No Calc:** $\lim_{x \to 1} \frac{\int_1^x e^{t^2}\,dt}{x^2 - 1}$ is

a. 0

b. 1

c. $\frac{e}{2}$

d. e

e. nonexistent

12. **2003 #80 (BC) - Calc OK:** Insects destroyed a crop at the rate of $\frac{100e^{-0.1t}}{2 - e^{-3t}}$ tons per day, where time t is measured in days. To the nearest ton, how many tons did the insects destroy during the time interval $7 \le t \le 14$?

a. 125

b. 100

c. 88

d. 50

e. 12

13. **1998 #88 (AB but suitable for BC) - Calc OK:** Let $F(x)$ be an antiderivative of $\frac{(\ln x)^3}{x}$. If $F(x) = 0$, then $F(9) =$

a. 0.048

b. 0.144

c. 5.827

d. 23.308

e. 1,640.250

Solutions:

1. D
2. E
3. A
4. E
5. B
6. B
7. C
8. E
9. D
10. C
11. C
12. A
13. C

Free Response Questions

❶ 2007 #3 (AB but suitable for BC) a,b,c – Calc OK

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	-4	4	2
4	-1	3	6	7

The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.

(a) Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.

(b) Explain why there must be a value c for $1 < c < 3$ such that $h'(c) = -5$.

(c) Let w be the function given by $w(x) = \int_1^{g(x)} f(t)\,dt$. Find the value of $w'(3)$.

❷ 2011B #5 (BC) c – No Calc

t (seconds)	0	10	40	60
$B(t)$ (meters)	100	136	9	49
$v(t)$ (meters per second)	2.0	2.3	2.5	4.6

Ben rides a unicycle back and forth along a straight east-west track. The twice-differentiable function B models Ben's position on the track, measured in meters from the western end of the track, at time t, measured in seconds from the start of the ride. The table above gives values for $B(t)$ and Ben's velocity, $v(t)$, measured in meters per second, at selected times t.

(c) For $40 \le t \le 60$, must there be a time t when Ben's velocity is 2 meters per second? Justify your answer.

❸ 2011B #1 (BC) d – Calc OK

A cylindrical can of radius 10 millimeters is used to measure rainfall in Stormville. The can is initially empty, and rain enters the can during a 60-day period. The height of water in the can is modeled by the function S, where $S(t)$ is measured in millimeters and t is measured in days for $0 \le t \le 60$. The rate at which the height of the water is rising in the can is given by $S'(t) = 2\sin(0.03t) + 1.5$.

(d) During the same 60-day period, rain on Monsoon Mountain accumulates in a can identical to the one in Stormville. The height of the water in the can on Monsoon Mountain is modeled by the function M, where $M(t) = \frac{1}{400}\left(3t^3 - 30t^2 + 330t\right)$. The height $M(t)$ is measured in millimeters, and t is measured in days for $0 \le t \le 60$. Let $D(t) = M'(t) - S'(t)$. Apply the Intermediate Value Theorem to the function D on the interval $0 \le t \le 60$ to justify that there exists a time t, $0 < t < 60$, at which the heights of water in the two cans are changing at the same rate.

❹ 2010 #1 (BC) c,d – Calc OK

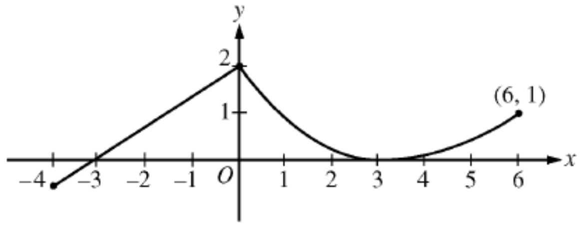
There is no snow on Janet's driveway when snow begins to fall at midnight. From midnight to 9 A.M., snow accumulates on the driveway at a rate modeled by $f(t) = 7te^{\cos t}$ cubic feet per hour, where t is measured in hours since midnight. Janet starts removing snow at 6 A.M. $(t = 6)$. The rate $g(t)$, in cubic feet per hour, at which Janet removes snow from the driveway at time t hours after midnight is modeled by

$$g(t) = \begin{cases} 0 & \text{for } 0 \le t < 6 \\ 125 & \text{for } 6 \le t < 7 \\ 108 & \text{for } 7 \le t \le 9. \end{cases}$$

(c) Let $h(t)$ represent the total amount of snow, in cubic feet, that Janet has removed from the driveway at time t hours after midnight. Express h as a piecewise-defined function with domain $0 \le t \le 9$.

(d) How many cubic feet of snow are on the driveway at 9 A.M.?

❺ 2009B #3 (BC) c – Calc OK

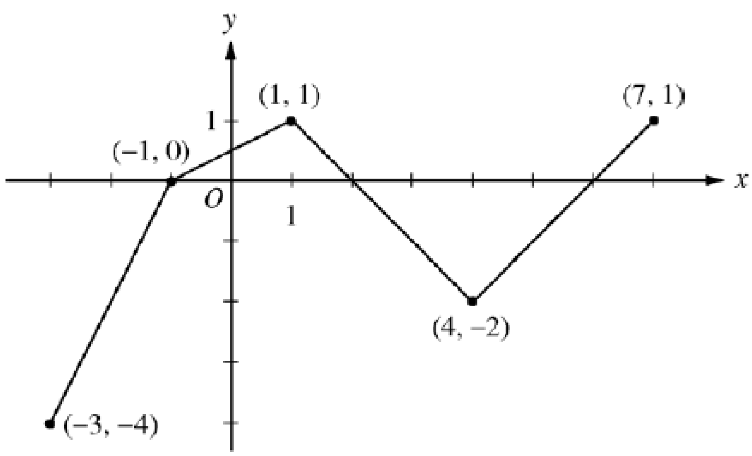


Graph of f

A continuous function f is defined on the closed interval $-4 \le x \le 6$. The graph of f consists of a line segment and a curve that is tangent to the x-axis at $x = 3$, as shown in the figure above. On the interval $0 < x < 6$, the function f is twice differentiable, with $f''(x) > 0$.

(c) Is there a value of a, $-4 \le a < 6$, for which the Mean Value Theorem, applied to the interval $[a, 6]$, guarantees a value c, $a < c < 6$, at which $f'(c) = \frac{1}{3}$? Justify your answer.

❻ 2008B #5 (BC) d – No Calc



Graph of g'

Let g be a continuous function with $g(2) = 5$. The graph of the piecewise-linear function g', the derivative of g, is shown above for $-3 \le x \le 7$.

(d) Find the average rate of change of $g'(x)$ on the interval $-3 \le x \le 7$. Does the Mean Value Theorem applied on the interval $-3 \le x \le 7$ guarantee a value of c, for $-3 < c < 7$, such that $g''(c)$ is equal to this average rate of change? Why or why not?

❼ 2009 #2 (BC) a,b,c – Calc OK

The rate at which people enter an auditorium for a rock concert is modeled by the function R given by $R(t) = 1380t^2 - 675t^3$ for $0 \le t \le 2$ hours; $R(t)$ is measured in people per hour. No one is in the auditorium at time $t = 0$, when the doors open. The doors close and the concert begins at time $t = 2$.

(a) How many people are in the auditorium when the concert begins?

(b) Find the time when the rate at which people enter the auditorium is a maximum. Justify your answer.

(c) The total wait time for all the people in the auditorium is found by adding the time each person waits, starting at the time the person enters the auditorium and ending when the concert begins. The function w models the total wait time for all the people who enter the auditorium before time t. The derivative of w is given by $w'(t) = (2 - t)R(t)$. Find $w(2) - w(1)$, the total wait time for those who enter the auditorium after time $t = 1$.

❶ 2007 #3 (AB but suitable for BC) a,b,c – Calc OK – Scoring Guidelines:

(a) $h(1) = f(g(1)) - 6 = f(2) - 6 = 9 - 6 = 3$

$h(3) = f(g(3)) - 6 = f(4) - 6 = -1 - 6 = -7$

Since $h(3) < -5 < h(1)$ and h is continuous, by the Intermediate Value Theorem, there exists a value r, $1 < r < 3$, such that $h(r) = -5$.

$2 : \begin{cases} 1 : h(1) \text{ and } h(3) \\ 1 : \text{conclusion, using IVT} \end{cases}$

(b) $\dfrac{h(3) - h(1)}{3 - 1} = \dfrac{-7 - 3}{3 - 1} = -5$

Since h is continuous and differentiable, by the Mean Value Theorem, there exists a value c, $1 < c < 3$, such that $h'(c) = -5$.

$2 : \begin{cases} 1 : \dfrac{h(3) - h(1)}{3 - 1} \\ 1 : \text{conclusion, using MVT} \end{cases}$

(c) $w'(3) = f(g(3)) \cdot g'(3) = f(4) \cdot 2 = -2$

$2 : \begin{cases} 1 : \text{apply chain rule} \\ 1 : \text{answer} \end{cases}$

❷ 2011B #5 (BC) c – No Calc – Scoring Guidelines:

(c) Because $\dfrac{B(60) - B(40)}{60 - 40} = \dfrac{49 - 9}{20} = 2$, the Mean Value Theorem implies there is a time t, $40 < t < 60$, such that $v(t) = 2$.

$2 : \begin{cases} 1 : \text{difference quotient} \\ 1 : \text{conclusion with justification} \end{cases}$

❸ 2011B #1 (BC) d – Calc OK – Scoring Guidelines:

(d) $D(0) = -0.675 < 0$ and $D(60) = 69.37730 > 0$

Because D is continuous, the Intermediate Value Theorem implies that there is a time t, $0 < t < 60$, at which $D(t) = 0$. At this time, the heights of water in the two cans are changing at the same rate.

$2 : \begin{cases} 1 : \text{considers } D(0) \text{ and } D(60) \\ 1 : \text{justification} \end{cases}$

❹ 2010 #1 (BC) c,d – Calc OK – Scoring Guidelines:

(c) $h(0) = 0$

For $0 < t \le 6$, $h(t) = h(0) + \int_0^t g(s)\,ds = 0 + \int_0^t 0\,ds = 0$.

For $6 < t \le 7$, $h(t) = h(6) + \int_6^t g(s)\,ds = 0 + \int_6^t 125\,ds = 125(t - 6)$.

For $7 < t \le 9$, $h(t) = h(7) + \int_7^t g(s)\,ds = 125 + \int_7^t 108\,ds = 125 + 108(t - 7)$.

Thus, $h(t) = \begin{cases} 0 & \text{for } 0 \le t \le 6 \\ 125(t - 6) & \text{for } 6 < t \le 7 \\ 125 + 108(t - 7) & \text{for } 7 < t \le 9 \end{cases}$

$3 : \begin{cases} 1 : h(t) \text{ for } 0 \le t \le 6 \\ 1 : h(t) \text{ for } 6 < t \le 7 \\ 1 : h(t) \text{ for } 7 < t \le 9 \end{cases}$

(d) Amount of snow is $\int_0^9 f(t)\,dt - h(9) = 26.334$ or 26.335 cubic feet.

$3 : \begin{cases} 1 : \text{integral} \\ 1 : h(9) \\ 1 : \text{answer} \end{cases}$

❺ 2009B #3 (BC) c – Calc OK – Scoring Guidelines:

(c) Yes, $a = 3$. The function f is differentiable on the interval $3 < x < 6$ and continuous on $3 \le x \le 6$. Also, $\frac{f(6) - f(3)}{6 - 3} = \frac{1 - 0}{6 - 3} = \frac{1}{3}$. By the Mean Value Theorem, there is a value c, $3 < c < 6$, such that $f'(c) = \frac{1}{3}$.	$2 : \begin{cases} 1 : \text{answers "yes" and identifies } a = 3 \\ 1 : \text{justification} \end{cases}$

❻ 2008B #5 (BC) d – No Calc – Scoring Guidelines:

(d) $\frac{g'(7) - g'(-3)}{7 - (-3)} = \frac{1 - (-4)}{10} = \frac{1}{2}$ No, the MVT does *not* guarantee the existence of a value c with the stated properties because g' is not differentiable for at least one point in $-3 < x < 7$.	$2 : \begin{cases} 1 : \text{ average value of } g'(x) \\ 1 : \text{ answer "No" with reason} \end{cases}$

❼ 2009 #2 (BC) a,b,c – Calc OK – Scoring Guidelines:

(a) $\int_0^2 R(t)\,dt = 980$ people	$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$
(b) $R'(t) = 0$ when $t = 0$ and $t = 1.36296$ The maximum rate may occur at 0, $a = 1.36296$, or 2. $R(0) = 0$ $R(a) = 854.527$ $R(2) = 120$ The maximum rate occurs when $t = 1.362$ or 1.363.	$3 : \begin{cases} 1 : \text{considers } R'(t) = 0 \\ 1 : \text{interior critical point} \\ 1 : \text{answer and justification} \end{cases}$
(c) $w(2) - w(1) = \int_1^2 w'(t)\,dt = \int_1^2 (2 - t)R(t)\,dt = 387.5$ The total wait time for those who enter the auditorium after time $t = 1$ is 387.5 hours.	$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

Looking for some more multiple choice problems to practice? Try these!

14. **1998 #15 (AB but suitable for BC) - No Calc:** If $F(x) = \int_0^x \sqrt{t^3 + 1}\, dt$, then $F'(2) =$

a. -3

b. -2

c. 2

d. 3

e. 18

15. **1997 #88 (BC) - Calc OK:** Let $f(x) = \int_0^{x^2} \sin t\, dt$. At how many points in the closed interval $[0, \sqrt{\pi}]$ does the instantaneous rate of change of f equal the average rate of change of f on that interval?

a. Zero

b. One

c. Two

d. Three

e. Four

16. **2003 #23 (AB but suitable for BC) - No Calc:** $\frac{d}{dx}\left(\int_0^{x^2} \sin\left(t^3\right) dt\right) =$

a. $-\cos\left(x^6\right)$

b. $\sin\left(x^3\right)$

c. $\sin\left(x^6\right)$

d. $2x\sin\left(x^3\right)$

e. $2x\sin\left(x^6\right)$

17. **2003 #92 (AB but suitable for BC) - Calc OK:** Let g be the function given by $g(x) = \int_0^x \sin\left(t^2\right) dt$ for $-1 \le x \le 3$. On which of the following intervals is g decreasing?

a. $-1 \le x \le 0$

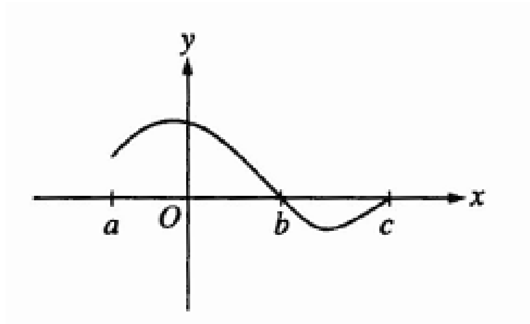
b. $0 \le x \le 1.772$

c. $1.253 \le x \le 2.171$

d. $1.772 \le x \le 2.507$

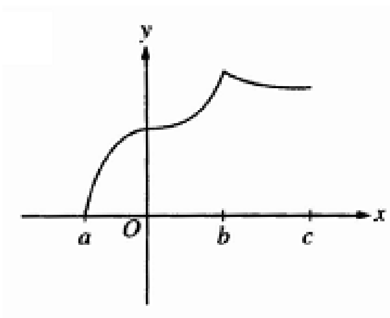
e. $2.802 \le x \le 3$

18. **1997 #88 (AB but suitable for BC) - Calc OK:**

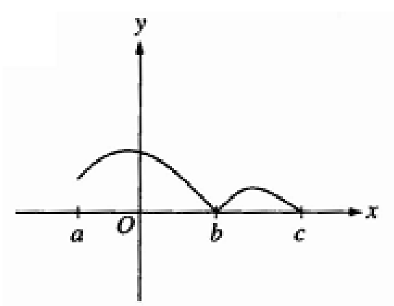


Let $f(x) = \int_a^x h(t)\,dt$, where h has the graph shown above. Which of the following could be the graph of f?

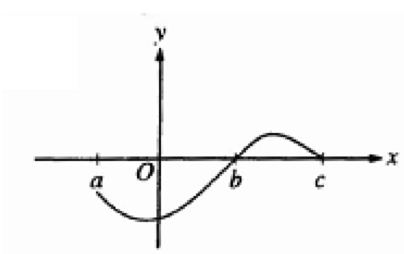
a.



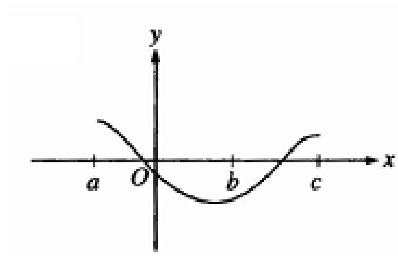
b.



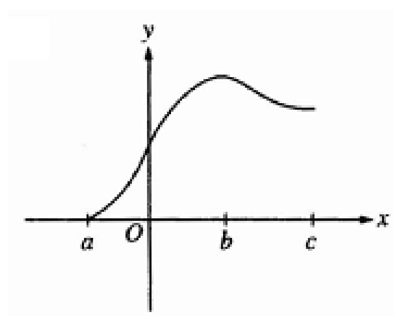
c.



d.



e.



Solutions:

14. D
15. C
16. E
17. D
18. E