AP Calculus BC Saturday Study Session #1: Relationships and Applications of f, f' and f''

(With special thanks to AdvanceKentucky)

The relationship between the graph of a function and its first and second derivatives frequently appears on the AP exams. It will appear on the free response section, often with the graph of f' given. Sometimes these questions utilize the Fundamental Theorem of Calculus by defining a function as a definite integral and providing information, usually a graph, of the integrand.

Most of these questions require you to justify your answers. A justification should make use of a known calculus "test" or theorem. You must show/state that the hypotheses are true and then draw the correct conclusion. Possible justifications include:

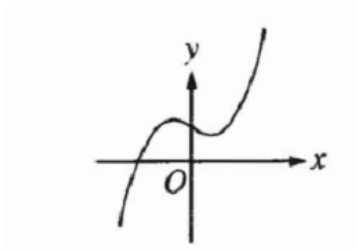
- If $f' > 0$ on an interval, then f is **increasing** on that interval.
- If $f' < 0$ on an interval, then f is **decreasing** on that interval.
- If $f'' > 0$ on an interval, then f is **concave up** on that interval.
- *Alternately*: If f' is increasing on an interval, then the graph of f is **concave up** on that interval.
- If $f'' < 0$ on an interval, then f is **concave down** on that interval.
- *Alternately:* If f' is decreasing on an interval, then the graph of f is **concave down** on that interval.
- If f changes from concave up to concave down (or vice versa) at $x = a$, then f has an **inflection point** at $x = a$.
- *Alternately:* If f'' changes from positive to negative (or vice versa) at $x = a$, then f has an **inflection point** at $x = a$.
- **The First Derivative Test**
  - If the sign of f' changes from positive to negative at $x = c$, then f has a relative (a.k.a, local) maximum at $x = c$.
  - If the sign of f' changes from negative to positive at $x = c$, then f has a relative (a.k.a, local) minimum at $x = c$.
- **The Second Derivative Test**
  - If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is a relative minimum.
  - If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is a relative maximum.
- **The First Fundamental Theorem of Calculus (1^{st} FTC)**
  - If $F(x)$ is an antiderivative of the continuous function $f(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$.
  - This can also be written as $\int_a^b f'(x)dx = f(b) - f(a)$ or $\int_a^x f'(t)dt = f(x) - f(a)$.
  - This last equation yields the incredibly useful formula $f(x) = f(a) + \int_a^x f'(t)dt$.
- **The Second Fundamental Theorem of Calculus (2^{nd} FTC)**
  - $\frac{d}{dx}\int_a^x f(t)dt = f(x)$
  - *Chain Rule Version:* $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = f(h(x)) \cdot h'(x) - f(g(x)) \cdot g'(x)$

Students need to be able to:

- Determine whether a function is increasing or decreasing using information about the derivative.
- Determine the concavity of a function's graph using information about the first or second derivative.
- Locate a function's relative and absolute extrema from its derivative.
- Locate a function's point(s) of inflection from its first or second derivative.
- Reason from a graph without finding an explicit rule that represents the graph.
- Write justifications and explanations.

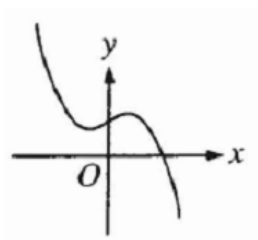
Multiple Choice Questions

1. **1998 #6 (BC) - No Calc:**

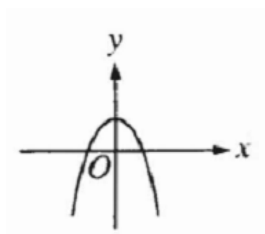


The graph of $y = h(x)$ is shown above. Which of the following could be the graph of $y = h'(x)$?

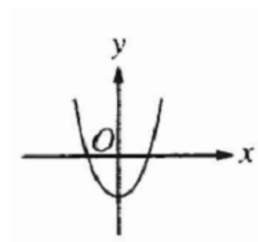
a.



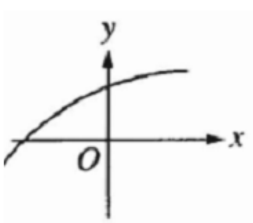
c.



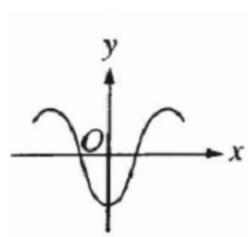
e.



b.



d.

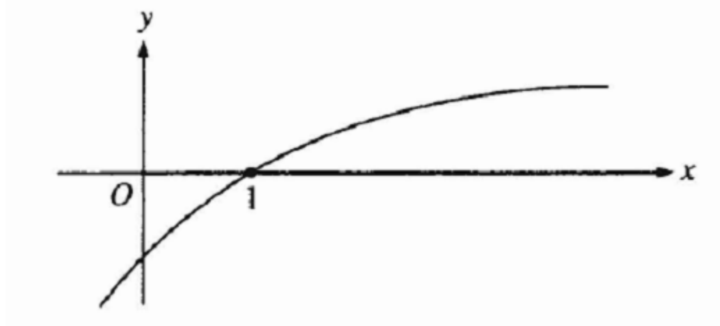


2. **1998 #1 (BC) - No Calc:** What are all values of x for which the function f defined by $f(x) = x^3 + 3x^2 - 9x + 7$ is increasing?

a. $-3 < x < 1$ c. $x < -3$ or $x > 1$ e. All real numbers

b. $-1 < x < 1$ d. $x < -1$ or $x > 3$

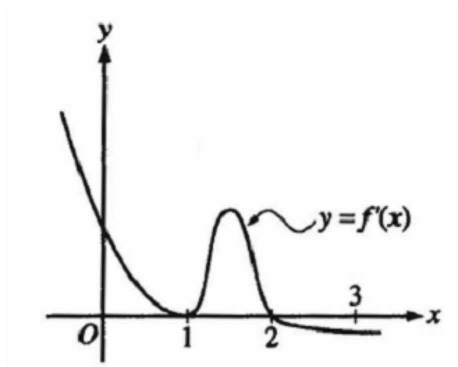
3. **1998 #17 (BC) - No Calc:**



The graph of a twice-differentiable function f is shown in the figure above. Which of the following is true?

a. $f(1) < f'(1) < f''(1)$
b. $f(1) < f''(1) < f'(1)$
c. $f'(1) < f(1) < f''(1)$
d. $f''(1) < f(1) < f'(1)$
e. $f''(1) < f'(1) < f(1)$

4. **2003 #90 (BC) - Calc OK:**

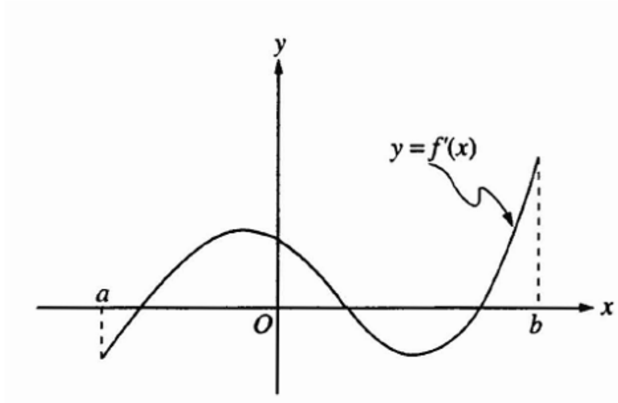


The graph of f', the derivative of the function f, is shown above. If $f(0) = 0$, which of the following must be true?

I. $f(0) > f(1)$
II. $f(2) > f(1)$
III. $f(1) > f(3)$

a. I only
b. II only
c. III only
d. I and II only
e. II and III only

5. **1997 #12 (BC) - No Calc:**



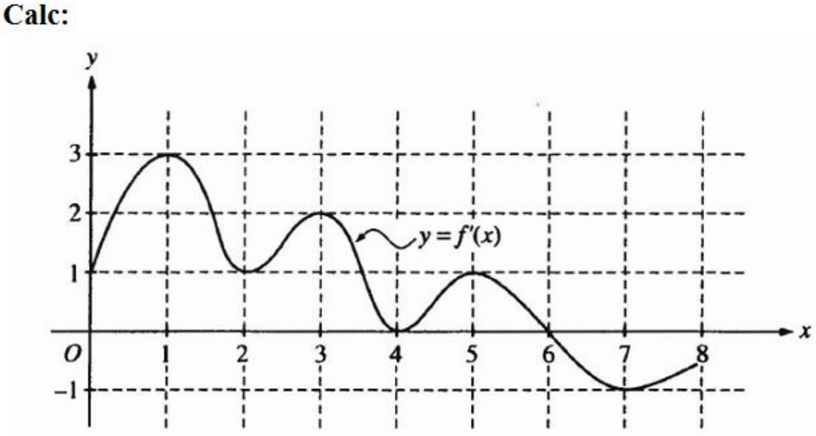
The graph of f', the derivative of f, is shown in the figure above. Which of the following describes all relative extrema of f on the open interval (a,b)?

a. One relative maximum and two relative minima
b. Two relative maxima and one relative minimum
c. Three relative maxima and one relative minimum
d. One relative maximum and three relative minima
e. Three relative maxima and two relative minima

6. **1997 #3 (BC) - No Calc:** The function f given by $f(x) = 3x^5 - 4x^3 - 3x$ has a relative maximum at $x =$

a. -1
b. $\frac{-\sqrt{5}}{5}$
c. 0
d. $\frac{\sqrt{5}}{5}$
e. 1

7. **1997 #9 (BC) - No Calc:**



The function f is defined on the closed interval $[0,8]$. The graph of its derivative f' is shown above. At what value of x does the absolute minimum of f occur?

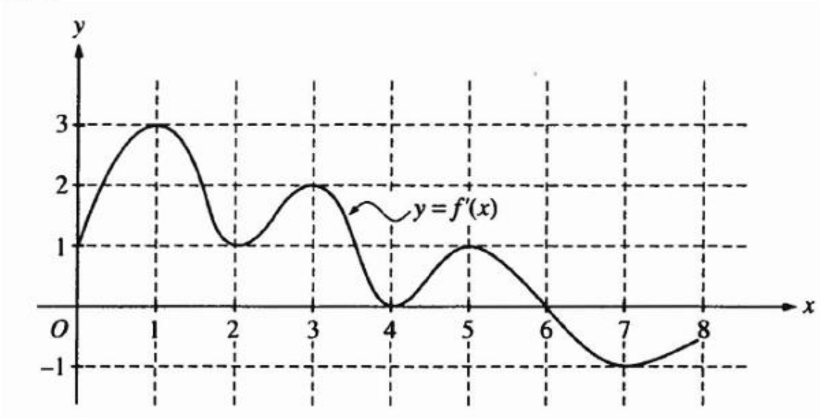
a. 0
b. 2
c. 4
d. 6
e. 8

8. **1998 #16 (BC) - No Calc:** If f is the function defined by $f(x) = 3x^5 - 5x^4$, what are all the x-coordinates of points of inflection for the graph of f?

a. -1 c. 1 e. -1, 0, and 1

b. 0 d. 0 and 1

9. **1997 #8 (BC) - No Calc:**



The function f is defined on the closed interval $[0,8]$. The graph of its derivative f' is shown above. How many points of inflection does the graph of f have?

a. Two b. Three c. Four d. Five e. Six

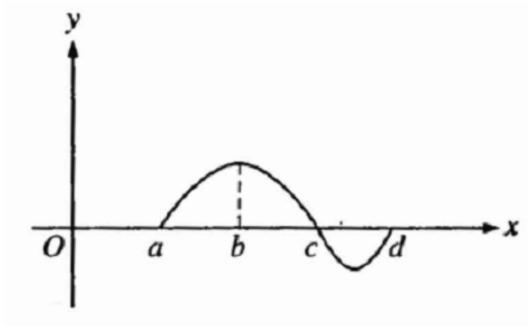
10. **2003 #86 (BC) - Calc OK:** Let f be the function with derivative defined by $f'(x) = \sin\left(x^3\right)$ on the interval $-1.8 < x < 1.8$. How many points of inflection does the graph of f have on this interval?

a. Two b. Three c. Four d. Five e. Six

11. **1997 #80 (BC) - Calc OK:** Let f be the function given by $f(x) = \cos(2x) + \ln(3x)$. What is the least value of x at which the graph of f changes concavity?

a. 0.56 b. 0.93 c. 1.18 d. 2.38 e. 2.44

12. **1997 #22 (BC) - No Calc:**



The graph of f is shown in the figure above. If $g(x)=\int_a^x f(t)\,dt$, for what value of x does $g(x)$ have a maximum?

a. a

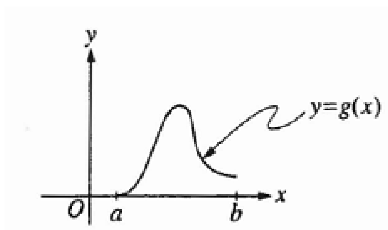
b. b

c. c

d. d

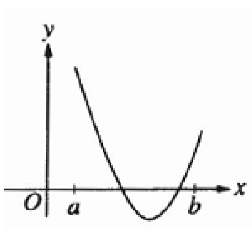
e. It cannot be determined from the information given.

13. **1998 #88 (BC) - Calc OK:**

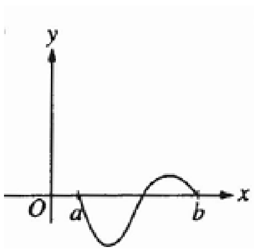


Let $g(x)=\int_a^x f(t)\,dt$, where $a \le x \le b$. The figure above shows the graph of g on $[a,b]$. Which of the following could be the graph of f on $[a,b]$?

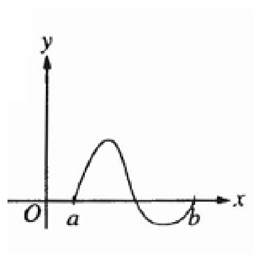
a.



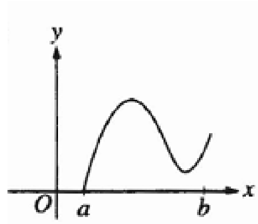
b.



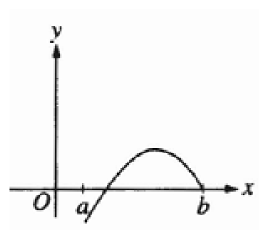
c.



d.



e.

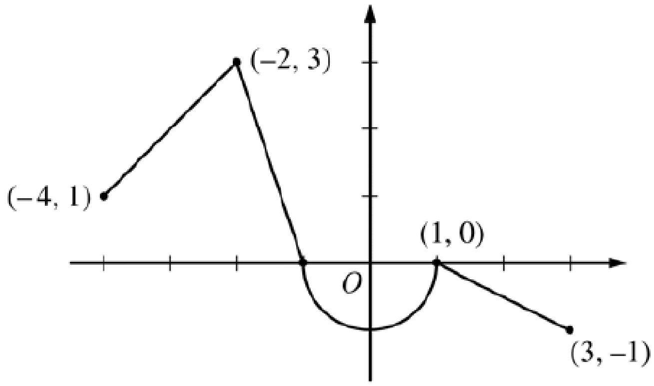


Solutions:

1. E
2. C
3. D
4. B
5. A
6. A
7. A
8. C
9. E
10. C
11. B
12. C
13. C

Free Response Questions

❶ 2012 #3 (AB & BC) – No Calc



Graph of f

Let f be the continuous function defined on $[-4, 3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given above. Let g be the function given by $g(x) = \int_1^x f(t)\,dt$.

(a) Find the values of $g(2)$ and $g(-2)$.

(b) For each of $g'(-3)$ and $g''(-3)$, find the value or state that it does not exist.

(c) Find the x-coordinate of each point at which the graph of g has a horizontal tangent line. For each of these points, determine whether g has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers.

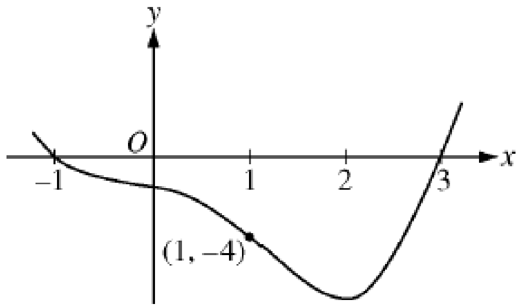
(d) For $-4 < x < 3$, find all values of x for which the graph of g has a point of inflection. Explain your reasoning.

❷ 2010B #5 (BC) a – No Calc

Let f and g be the functions defined by $f(x) = \frac{1}{x}$ and $g(x) = \frac{4x}{1 + 4x^2}$, for all $x > 0$.

(a) Find the absolute maximum value of g on the open interval $(0, \infty)$ if the maximum exists. Find the absolute minimum value of g on the open interval $(0, \infty)$ if the minimum exists. Justify your answers.

❸ 2009B #5 (AB & BC) a,b,c – No Calc



Graph of f'

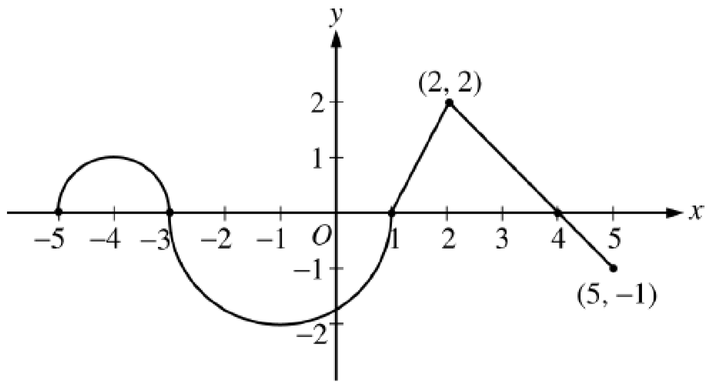
Let f be a twice-differentiable function defined on the interval $-1.2 < x < 3.2$ with $f(1) = 2$. The graph of f', the derivative of f, is shown above. The graph of f' crosses the x-axis at $x = -1$ and $x = 3$ and has a horizontal tangent at $x = 2$. Let g be the function given by $g(x) = e^{f(x)}$.

(a) Write an equation for the line tangent to the graph of g at $x = 1$.

(b) For $-1.2 < x < 3.2$, find all values of x at which g has a local maximum. Justify your answer.

(c) The second derivative of g is $g''(x) = e^{f(x)}\left[(f'(x))^2 + f''(x)\right]$. Is $g''(-1)$ positive, negative, or zero? Justify your answer.

❹ 2007B #4 (AB & BC) – No Calc



Graph of f'

Let f be a function defined on the closed interval $-5 \le x \le 5$ with $f(1) = 3$. The graph of f', the derivative of f, consists of two semicircles and two line segments, as shown above.

(a) For $-5 < x < 5$, find all values x at which f has a relative maximum. Justify your answer.

(b) For $-5 < x < 5$, find all values x at which the graph of f has a point of inflection. Justify your answer.

(c) Find all intervals on which the graph of f is concave up and also has positive slope. Explain your reasoning.

(d) Find the absolute minimum value of $f(x)$ over the closed interval $-5 \le x \le 5$. Explain your reasoning.

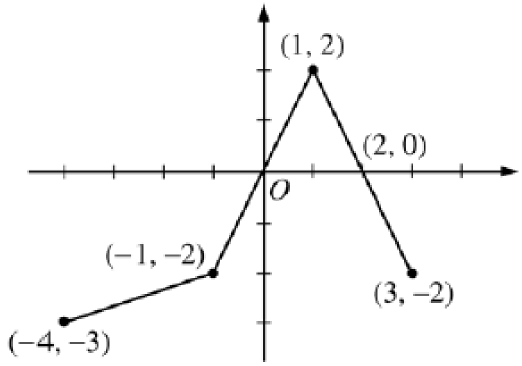
❺ 2008 #5 (BC) a,b – No Calc

The derivative of a function f is given by $f'(x) = (x - 3)e^x$ for $x > 0$, and $f(1) = 7$.

(a) The function f has a critical point at $x = 3$. At this point, does f have a relative minimum, a relative maximum, or neither? Justify your answer.

(b) On what intervals, if any, is the graph of f both decreasing and concave up? Explain your reasoning.

❻ 2005B #4 (AB & BC) – No Calc



Graph of f

The graph of the function f above consists of three line segments.

(a) Let g be the function given by $g(x) = \int_{-4}^{x} f(t)\,dt$. For each of $g(-1)$, $g'(-1)$, and $g''(-1)$, find the value or state that it does not exist.

(b) For the function g defined in part (a), find the x-coordinate of each point of inflection of the graph of g on the open interval $-4 < x < 3$. Explain your reasoning.

(c) Let h be the function given by $h(x) = \int_{x}^{3} f(t)\,dt$. Find all values of x in the closed interval $-4 \le x \le 3$ for which $h(x) = 0$.

(d) For the function h defined in part (c), find all intervals on which h is decreasing. Explain your reasoning.

❼ 2004 #4 (AB & BC) a,c – No Calc

Consider the curve given by $x^2 + 4y^2 = 7 + 3xy$.

(a) Show that $\frac{dy}{dx} = \frac{3y - 2x}{8y - 3x}$.

(c) Find the value of $\frac{d^2y}{dx^2}$ at the point P found in part (b). Does the curve have a local maximum, a local minimum, or neither at the point P ? Justify your answer.

❶ 2012 #3 (AB & BC) – No Calc – Scoring Guidelines:

(a) $g(2) = \int_1^2 f(t)\,dt = -\frac{1}{2}(1)\left(\frac{1}{2}\right) = -\frac{1}{4}$

$g(-2) = \int_1^{-2} f(t)\,dt = -\int_{-2}^{1} f(t)\,dt$

$= -\left(\frac{3}{2} - \frac{\pi}{2}\right) = \frac{\pi}{2} - \frac{3}{2}$

$2 : \begin{cases} 1 : g(2) \\ 1 : g(-2) \end{cases}$

(b) $g'(x) = f(x) \;\Rightarrow\; g'(-3) = f(-3) = 2$

$g''(x) = f'(x) \;\Rightarrow\; g''(-3) = f'(-3) = 1$

$2 : \begin{cases} 1 : g'(-3) \\ 1 : g''(-3) \end{cases}$

(c) The graph of g has a horizontal tangent line where $g'(x) = f(x) = 0$. This occurs at $x = -1$ and $x = 1$.

$g'(x)$ changes sign from positive to negative at $x = -1$. Therefore, g has a relative maximum at $x = -1$.

$g'(x)$ does not change sign at $x = 1$. Therefore, g has neither a relative maximum nor a relative minimum at $x = 1$.

$3 : \begin{cases} 1 : \text{considers } g'(x) = 0 \\ 1 : x = -1 \text{ and } x = 1 \\ 1 : \text{answers with justifications} \end{cases}$

(d) The graph of g has a point of inflection at each of $x = -2$, $x = 0$, and $x = 1$ because $g''(x) = f'(x)$ changes sign at each of these values.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{explanation} \end{cases}$

❷ 2010B #5 (BC) a – No Calc – Scoring Guidelines:

(a) $g'(x) = \dfrac{4\left(1 + 4x^2\right) - 4x(8x)}{\left(1 + 4x^2\right)^2} = \dfrac{4\left(1 - 4x^2\right)}{\left(1 + 4x^2\right)^2}$

For $x > 0$, $g'(x) = 0$ for $x = \frac{1}{2}$.

$g'(x) > 0$ for $0 < x < \frac{1}{2}$

$g'(x) < 0$ for $x > \frac{1}{2}$

$g\left(\frac{1}{2}\right) = 1$

Therefore g has a maximum value of 1 at $x = \frac{1}{2}$, and g has no minimum value on the open interval $(0, \infty)$.

$5 : \begin{cases} 2 : g'(x) \\ 1 : \text{critical point} \\ 1 : \text{answers} \\ 1 : \text{justification} \end{cases}$

❸ 2009B #5 (AB & BC) a,b,c – No Calc – Scoring Guidelines:

(a) $g(1) = e^{f(1)} = e^2$

$g'(x) = e^{f(x)} f'(x)$, $g'(1) = e^{f(1)} f'(1) = -4e^2$

The tangent line is given by $y = e^2 - 4e^2(x-1)$.

$3 : \begin{cases} 1 : g'(x) \\ 1 : g(1) \text{ and } g'(1) \\ 1 : \text{tangent line equation} \end{cases}$

(b) $g'(x) = e^{f(x)} f'(x)$

$e^{f(x)} > 0$ for all x

So, g' changes from positive to negative only when f' changes from positive to negative. This occurs at $x = -1$ only. Thus, g has a local maximum at $x = -1$.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

(c) $g''(-1) = e^{f(-1)} \left[(f'(-1))^2 + f''(-1) \right]$

$e^{f(-1)} > 0$ and $f'(-1) = 0$

Since f' is decreasing on a neighborhood of -1, $f''(-1) < 0$. Therefore, $g''(-1) < 0$.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

❹ 2007B #4 (AB & BC) – No Calc – Scoring Guidelines:

(a) $f'(x) = 0$ at $x = -3,\ 1,\ 4$
f' changes from positive to negative at -3 and 4.
Thus, f has a relative maximum at $x = -3$ and at $x = 4$.

$2 : \begin{cases} 1 : x\text{-values} \\ 1 : \text{justification} \end{cases}$

(b) f' changes from increasing to decreasing, or vice versa, at $x = -4,\ -1,$ and 2. Thus, the graph of f has points of inflection when $x = -4,\ -1,$ and 2.

$2 : \begin{cases} 1 : x\text{-values} \\ 1 : \text{justification} \end{cases}$

(c) The graph of f is concave up with positive slope where f' is increasing and positive: $-5 < x < -4$ and $1 < x < 2$.

$2 : \begin{cases} 1 : \text{intervals} \\ 1 : \text{explanation} \end{cases}$

(d) Candidates for the absolute minimum are where f' changes from negative to positive (at $x = 1$) and at the endpoints ($x = -5, 5$).

$$f(-5) = 3 + \int_1^{-5} f'(x)\,dx = 3 - \frac{\pi}{2} + 2\pi > 3$$

$$f(1) = 3$$

$$f(5) = 3 + \int_1^{5} f'(x)\,dx = 3 + \frac{3 \cdot 2}{2} - \frac{1}{2} > 3$$

The absolute minimum value of f on $[-5, 5]$ is $f(1) = 3$.

$3 : \begin{cases} 1 : \text{identifies } x = 1 \text{ as a candidate} \\ 1 : \text{considers endpoints} \\ 1 : \text{value and explanation} \end{cases}$

❺ 2008 #5 (BC) a,b – No Calc – Scoring Guidelines:

(a) $f'(x) < 0$ for $0 < x < 3$ and $f'(x) > 0$ for $x > 3$

Therefore, f has a relative minimum at $x = 3$.

$2 : \begin{cases} 1\text{: minimum at } x = 3 \\ 1\text{: justification} \end{cases}$

(b) $f''(x) = e^x + (x-3)e^x = (x-2)e^x$
$f''(x) > 0$ for $x > 2$

$f'(x) < 0$ for $0 < x < 3$

Therefore, the graph of f is both decreasing and concave up on the interval $2 < x < 3$.

$3 : \begin{cases} 2 : f''(x) \\ 1 : \text{answer with reason} \end{cases}$

❻ 2005B #4 (AB & BC) – No Calc – Scoring Guidelines:

(a) $g(-1) = \int_{-4}^{-1} f(t)\,dt = -\frac{1}{2}(3)(5) = -\frac{15}{2}$ $g'(-1) = f(-1) = -2$ $g''(-1)$ does not exist because f is not differentiable at $x = -1$.	$3 : \begin{cases} 1 : g(-1) \\ 1 : g'(-1) \\ 1 : g''(-1) \end{cases}$
(b) $x = 1$ $g' = f$ changes from increasing to decreasing at $x = 1$.	$2 : \begin{cases} 1 : x = 1 \text{ (only)} \\ 1 : \text{reason} \end{cases}$
(c) $x = -1,\ 1,\ 3$	2 : correct values $\langle -1 \rangle$ each missing or extra value
(d) h is decreasing on $[0, 2]$ $h' = -f < 0$ when $f > 0$	$2 : \begin{cases} 1 : \text{interval} \\ 1 : \text{reason} \end{cases}$

❼ 2004 #4 (AB & BC) a,c – No Calc – Scoring Guidelines:

(a) $2x + 8yy' = 3y + 3xy'$ $(8y - 3x)y' = 3y - 2x$ $y' = \frac{3y - 2x}{8y - 3x}$	$2 : \begin{cases} 1 : \text{implicit differentiation} \\ 1 : \text{solves for } y' \end{cases}$
(c) $\frac{d^2y}{dx^2} = \frac{(8y-3x)(3y'-2) - (3y-2x)(8y'-3)}{(8y-3x)^2}$ At $P = (3, 2)$, $\frac{d^2y}{dx^2} = \frac{(16-9)(-2)}{(16-9)^2} = -\frac{2}{7}$. Since $y' = 0$ and $y'' < 0$ at P, the curve has a local maximum at P.	$4 : \begin{cases} 2 : \frac{d^2y}{dx^2} \\ 1 : \text{value of } \frac{d^2y}{dx^2} \text{ at } (3, 2) \\ 1 : \text{conclusion with justification} \end{cases}$