

AP Calculus BC Saturday Study Session #1: Computing Derivatives

(With special thanks to Lin McMullin, AdvanceKentucky & Amy Johnson-Lambert)

It goes without saying that derivatives are an important part of the calculus and you need to be able to compute them. You should know the derivatives of all the functions you've been studying:

- Polynomials
- Rational functions (quotients) and functions with radicals
- Trig functions
- Inverse trig functions (by implicit differentiation)
- Exponential and logarithmic functions

The AP exams will ask you to find derivatives using the various techniques and rules including:

- The **Power Rule** for integer, rational (fractional) exponents, expressions with radicals.
- Derivatives of sum, differences, **products**, and **quotients**.
- The **Chain Rule** for composite functions. Nearly every multiple-choice question on differentiation from past released exams uses the Chain Rule.
- **Implicit differentiation** which often shows up on multiple-choice and is sometimes an entire free-response question. These are often no calculator questions. Students should be able to:
 - Differentiate an implicit relation.
 - Verify the derivative of an implicit relation.
 - Write an equation for a line tangent to the graph of an implicit relation at a particular point.
 - Find the second derivative of an implicit relation.
 - Find coordinates of points at which an implicit relation has horizontal and/or vertical tangent lines.
 - Solve related rates problems.
- Find the derivative of expression in functional form (for example, $f(g(x))$ or $f(x) \cdot g(x)$) where the functions are not explicitly given and values are taken from a table or graph.

The questions on the exams are not overly complicated (no “monster” functions) and the aim is to be sure that you know the basics. The questions on the no calculator sections are straightforward.

Multiple Choice Questions

1. **2003 #1 (AB but suitable for BC) - No Calc:** If $y = (x^3 + 1)^2$, then $\frac{dy}{dx} =$
- a. $(3x^2)^2$ c. $2(3x^3 + 1)$ e. $6x^2(x^3 + 1)$
- b. $2(x^3 + 1)$ d. $3x^2(x^3 + 1)$
2. **2003 #9 (AB & BC) - No Calc:** If $f(x) = \ln(x + 4 + e^{-3x})$, then $f'(0)$ is
- a. $-\frac{2}{5}$ c. $\frac{1}{4}$ e. nonexistent
- b. $\frac{1}{5}$ d. $\frac{2}{5}$
3. **2003 #1 (BC) - Calc OK:** If $y = \sin(3x)$, then $\frac{dy}{dx} =$
- a. $-3 \cos(3x)$ c. $-\frac{1}{3} \cos(3x)$ e. $3 \cos(3x)$
- b. $-\cos(3x)$ d. $\cos(3x)$
4. **1997 #4 (BC) - No Calc:** $\frac{d}{dx} \left(x e^{\ln x^2} \right) =$
- a. $1 + 2x$ c. $3x^2$ e. $x^2 + x^3$
- b. $x + x^2$ d. x^3
5. **1997 #5 (BC) - No Calc:** If $f(x) = (x - 1)^{\frac{3}{2}} + \frac{e^{x-2}}{2}$, then $f'(2) =$
- a. 1 c. 2 e. $\frac{3+e}{2}$
- b. $\frac{3}{2}$ d. $\frac{7}{2}$

6. 1998 #28 (AB but suitable for BC) - No Calc: If $f(x) = \tan(2x)$, then $f'\left(\frac{\pi}{6}\right) =$

- a. $\sqrt{3}$ c. 4 e. 8
b. $2\sqrt{3}$ d. $4\sqrt{3}$

7. 2003 #14 (AB but suitable for BC) - No Calc: If $y = x^2 \sin 2x$, then $\frac{dy}{dx} =$

- a. $2x \cos 2x$ c. $2x(\sin 2x + \cos 2x)$ e. $2x(\sin 2x + x \cos 2x)$
b. $4x \cos 2x$ d. $2x(\sin 2x - x \cos 2x)$

8. 2003 #4 (AB but suitable for BC) - No Calc: If $y = \frac{2x+3}{3x+2}$, then $\frac{dy}{dx} =$

- a. $\frac{12x+13}{(3x+2)^2}$ c. $\frac{5}{(3x+2)^2}$ e. $\frac{2}{3}$
b. $\frac{12x-13}{(3x+2)^2}$ d. $\frac{-5}{(3x+2)^2}$

9. 1997 #2 (AB but suitable for BC) - No Calc: If $f(x) = x\sqrt{2x-3}$, then $f'(x) =$

- a. $\frac{3x-3}{\sqrt{2x-3}}$ c. $\frac{1}{\sqrt{2x-3}}$ e. $\frac{5x-6}{2\sqrt{2x-3}}$
b. $\frac{x}{\sqrt{2x-3}}$ d. $\frac{-x+3}{\sqrt{2x-3}}$

10. 2003 #26 (AB but suitable for BC) - No Calc: What is the slope of the line tangent to the curve $3y^2 - 2x^2 = 6 - 2xy$ at the point $(3, 2)$?

- a. 0 c. $\frac{7}{9}$ e. $\frac{5}{3}$
b. $\frac{4}{9}$ d. $\frac{6}{7}$

11. 1998 #3 (BC) - No Calc: The slope of the line tangent to the curve $y^2 + (xy + 1)^3 = 0$ at $(2, -1)$ is

a. $\frac{-3}{2}$

c. 0

e. $\frac{3}{2}$

b. $\frac{-3}{4}$

d. $\frac{3}{4}$

12. 1997 #10 (BC) - No Calc: If $y = xy + x^2 + 1$, then when $x = -1$, $\frac{dy}{dx}$ is

a. $\frac{1}{2}$

c. -1

e. nonexistent

b. $\frac{-1}{2}$

d. -2

13. 1998 #81 (BC) - Calc OK: If $\frac{dy}{dx} = \sqrt{1-y^2}$, then $\frac{d^2y}{dx^2} =$

a. $-2y$

c. $\frac{-y}{\sqrt{1-y^2}}$

e. $\frac{1}{2}$

b. $-y$

d. y

14. 1997 #17 (AB but suitable for BC) - No Calc: If $x^2 + y^2 = 25$, what is the value of $\frac{d^2y}{dx^2}$ at the point $(4, 3)$?

a. $\frac{-25}{27}$

c. $\frac{7}{27}$

e. $\frac{25}{27}$

b. $\frac{-7}{27}$

d. $\frac{3}{4}$

15. 2003 #79 (BC) - Calc OK:

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-1	6	5	3	-2
1	3	-3	-1	2
3	1	-2	2	3

The table above gives values of f , f' , g and g' at selected values of x . If $h(x) = f(g(x))$, then $h'(1) =$

- a. 5 b. 6 c. 9 d. 10 e. 12

16. 1998 #5 (BC) - No Calc: If f and g are twice differentiable and if $h(x) = f(g(x))$, then $h''(x) =$

- a. $f''(g(x))[g'(x)]^2 + f'(g(x))g''(x)$ d. $f''(g(x))g''(x)$
 b. $f''(g(x))g'(x) + f'(g(x))g''(x)$ e. $f''(g(x))$
 c. $f''(g(x))[g'(x)]^2$

17. 1997 #81 (AB but suitable for BC) - Calc OK: A railroad track and a road cross at right angles. An observer stands on the road 70 meters south of the crossing and watches an eastbound train traveling at 60 meters per second. At how many meters per second is the train moving away from the observer 4 seconds after it passes through the intersection?

- a. 57.60 c. 59.20 e. 67.40
 b. 57.88 d. 60.00

Solutions:

1. E
2. A
3. E
4. C
5. C
6. E
7. E
8. D
9. A
10. B
11. D
12. B
13. B
14. A
15. D
16. A
17. A

Free Response Questions

1 2010B #5 (BC) – No Calc

Let f and g be the functions defined by $f(x) = \frac{1}{x}$ and $g(x) = \frac{4x}{1 + 4x^2}$, for all $x > 0$.

- (a) Find the absolute maximum value of g on the open interval $(0, \infty)$ if the maximum exists. Find the absolute minimum value of g on the open interval $(0, \infty)$ if the minimum exists. Justify your answers.

2 2006 #6 (AB but suitable for BC) – No Calc

The twice-differentiable function f is defined for all real numbers and satisfies the following conditions:

$$f(0) = 2, \quad f'(0) = -4, \quad \text{and} \quad f''(0) = 3.$$

- (a) The function g is given by $g(x) = e^{ax} + f(x)$ for all real numbers, where a is a constant. Find $g'(0)$ and $g''(0)$ in terms of a . Show the work that leads to your answers.
- (b) The function h is given by $h(x) = \cos(kx)f(x)$ for all real numbers, where k is a constant. Find $h'(x)$ and write an equation for the line tangent to the graph of h at $x = 0$.

3 2005 B #5 (AB & BC) – No Calc

Consider the curve given by $y^2 = 2 + xy$.

- (a) Show that $\frac{dy}{dx} = \frac{y}{2y - x}$.
- (b) Find all points (x, y) on the curve where the line tangent to the curve has slope $\frac{1}{2}$.
- (c) Show that there are no points (x, y) on the curve where the line tangent to the curve is horizontal.
- (d) Let x and y be functions of time t that are related by the equation $y^2 = 2 + xy$. At time $t = 5$, the value of y is 3 and $\frac{dy}{dt} = 6$. Find the value of $\frac{dx}{dt}$ at time $t = 5$.

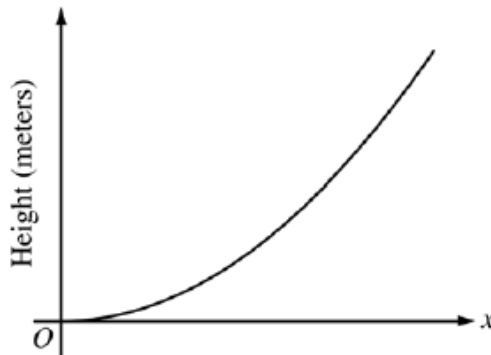
4 2008B #6 (AB but suitable for BC) – No Calc

Consider the closed curve in the xy -plane given by

$$x^2 + 2x + y^4 + 4y = 5.$$

- (a) Show that $\frac{dy}{dx} = \frac{-(x+1)}{2(y^3+1)}$.
- (b) Write an equation for the line tangent to the curve at the point $(-2, 1)$.
- (c) Find the coordinates of the two points on the curve where the line tangent to the curve is vertical.
- (d) Is it possible for this curve to have a horizontal tangent at points where it intersects the x -axis?
Explain your reasoning.

5 2006B #3 (BC) – Calc OK



The figure above is the graph of a function of x , which models the height of a skateboard ramp. The function meets the following requirements.

- (i) At $x = 0$, the value of the function is 0, and the slope of the graph of the function is 0.
 - (ii) At $x = 4$, the value of the function is 1, and the slope of the graph of the function is 1.
 - (iii) Between $x = 0$ and $x = 4$, the function is increasing.
- (a) Let $f(x) = ax^2$, where a is a nonzero constant. Show that it is not possible to find a value for a so that f meets requirement (ii) above.
- (b) Let $g(x) = cx^3 - \frac{x^2}{16}$, where c is a nonzero constant. Find the value of c so that g meets requirement (ii) above. Show the work that leads to your answer.
- (c) Using the function g and your value of c from part (b), show that g does not meet requirement (iii) above.
- (d) Let $h(x) = \frac{x^n}{k}$, where k is a nonzero constant and n is a positive integer. Find the values of k and n so that h meets requirement (ii) above. Show that h also meets requirements (i) and (iii) above.

6 2012 #4 (AB but suitable for BC) a, b – No Calc

The function f is defined by $f(x) = \sqrt{25 - x^2}$ for $-5 \leq x \leq 5$.

(a) Find $f'(x)$.

(b) Write an equation for the line tangent to the graph of f at $x = -3$.

1 2010B #5 (BC) a – No Calc – Scoring Guidelines:

$$g'(x) = \frac{4(1+4x^2) - 4x(8x)}{(1+4x^2)^2} = \frac{4(1-4x^2)}{(1+4x^2)^2}$$

For $x > 0$, $g'(x) = 0$ for $x = \frac{1}{2}$.

$$g'(x) > 0 \text{ for } 0 < x < \frac{1}{2}$$

$$g'(x) < 0 \text{ for } x > \frac{1}{2}$$

$$g\left(\frac{1}{2}\right) = 1$$

Therefore g has a maximum value of 1 at $x = \frac{1}{2}$, and g has no minimum value on the open interval $(0, \infty)$.

5 : $\left\{ \begin{array}{l} 2 : g'(x) \\ 1 : \text{critical point} \\ 1 : \text{answers} \\ 1 : \text{justification} \end{array} \right.$

2 2006 #6 (AB but suitable for BC) – No Calc – Scoring Guidelines:

(a) $g'(x) = ae^{ax} + f'(x)$
 $g'(0) = a - 4$

$$g''(x) = a^2e^{ax} + f''(x)$$
$$g''(0) = a^2 + 3$$

(b) $h'(x) = f'(x)\cos(kx) - k\sin(kx)f(x)$
 $h'(0) = f'(0)\cos(0) - k\sin(0)f(0) = f'(0) = -4$
 $h(0) = \cos(0)f(0) = 2$
The equation of the tangent line is $y = -4x + 2$.

4 : $\left\{ \begin{array}{l} 1 : g'(x) \\ 1 : g'(0) \\ 1 : g''(x) \\ 1 : g''(0) \end{array} \right.$

5 : $\left\{ \begin{array}{l} 2 : h'(x) \\ 3 : \left\{ \begin{array}{l} 1 : h'(0) \\ 1 : h(0) \\ 1 : \text{equation of tangent line} \end{array} \right. \end{array} \right.$

3 2005 B #5 (AB & BC) – No Calc – Scoring Guidelines:

(a) $2yy' = y + xy'$
 $(2y - x)y' = y$
 $y' = \frac{y}{2y - x}$

$$2 : \begin{cases} 1 : \text{implicit differentiation} \\ 1 : \text{solves for } y' \end{cases}$$

(b) $\frac{y}{2y - x} = \frac{1}{2}$
 $2y = 2y - x$
 $x = 0$
 $y = \pm\sqrt{2}$
 $(0, \sqrt{2}), (0, -\sqrt{2})$

$$2 : \begin{cases} 1 : \frac{y}{2y - x} = \frac{1}{2} \\ 1 : \text{answer} \end{cases}$$

(c) $\frac{y}{2y - x} = 0$
 $y = 0$
 The curve has no horizontal tangent since
 $0^2 \neq 2 + x \cdot 0$ for any x .

$$2 : \begin{cases} 1 : y = 0 \\ 1 : \text{explanation} \end{cases}$$

(d) When $y = 3$, $3^2 = 2 + 3x$ so $x = \frac{7}{3}$.
 $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{y}{2y - x} \cdot \frac{dx}{dt}$
 At $t = 5$, $6 = \frac{3}{6 - \frac{7}{3}} \cdot \frac{dx}{dt} = \frac{9}{11} \cdot \frac{dx}{dt}$
 $\frac{dx}{dt} \Big|_{t=5} = \frac{22}{3}$

$$3 : \begin{cases} 1 : \text{solves for } x \\ 1 : \text{chain rule} \\ 1 : \text{answer} \end{cases}$$

4 2008 B (AB but suitable for BC) #6 – No Calc – Scoring Guidelines:

(a) $2x + 2 + 4y^3 \frac{dy}{dx} + 4 \frac{dy}{dx} = 0$

$$(4y^3 + 4) \frac{dy}{dx} = -2x - 2$$

$$\frac{dy}{dx} = \frac{-2(x+1)}{4(y^3+1)} = \frac{-(x+1)}{2(y^3+1)}$$

2 : $\begin{cases} 1 : \text{implicit differentiation} \\ 1 : \text{verification} \end{cases}$

(b) $\left. \frac{dy}{dx} \right|_{(-2,1)} = \frac{-(-2+1)}{2(1+1)} = \frac{1}{4}$

Tangent line: $y = 1 + \frac{1}{4}(x + 2)$

2 : $\begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line equation} \end{cases}$

(c) Vertical tangent lines occur at points on the curve where $y^3 + 1 = 0$ (or $y = -1$) and $x \neq -1$.

On the curve, $y = -1$ implies that $x^2 + 2x + 1 - 4 = 5$, so $x = -4$ or $x = 2$.

Vertical tangent lines occur at the points $(-4, -1)$ and $(2, -1)$.

3 : $\begin{cases} 1 : y = -1 \\ 1 : \text{substitutes } y = -1 \text{ into the} \\ \quad \text{equation of the curve} \\ 1 : \text{answer} \end{cases}$

(d) Horizontal tangents occur at points on the curve where $x = -1$ and $y \neq -1$.

The curve crosses the x -axis where $y = 0$.

$$(-1)^2 + 2(-1) + 0^4 + 4 \cdot 0 \neq 5$$

No, the curve cannot have a horizontal tangent where it crosses the x -axis.

2 : $\begin{cases} 1 : \text{works with } x = -1 \text{ or } y = 0 \\ 1 : \text{answer with reason} \end{cases}$

5 2006B #3 (BC) – Calc OK – Scoring Guidelines:

(a) $f(4) = 1$ implies that $a = \frac{1}{16}$ and $f'(4) = 2a(4) = 1$
implies that $a = \frac{1}{8}$. Thus, f cannot satisfy (ii).

2 : $\left\{ \begin{array}{l} 1 : a = \frac{1}{16} \text{ or } a = \frac{1}{8} \\ 1 : \text{shows } a \text{ does not work} \end{array} \right.$

(b) $g(4) = 64c - 1 = 1$ implies that $c = \frac{1}{32}$.
When $c = \frac{1}{32}$, $g'(4) = 3c(4)^2 - \frac{2(4)}{16} = 3\left(\frac{1}{32}\right)(16) - \frac{1}{2} = 1$

1 : value of c

(c) $g'(x) = \frac{3}{32}x^2 - \frac{x}{8} = \frac{1}{32}x(3x - 4)$
 $g'(x) < 0$ for $0 < x < \frac{4}{3}$, so g does not satisfy (iii).

2 : $\left\{ \begin{array}{l} 1 : g'(x) \\ 1 : \text{explanation} \end{array} \right.$

(d) $h(4) = \frac{4^n}{k} = 1$ implies that $4^n = k$.
 $h'(4) = \frac{n4^{n-1}}{k} = \frac{n4^{n-1}}{4^n} = \frac{n}{4} = 1$ gives $n = 4$ and $k = 4^4 = 256$.

4 : $\left\{ \begin{array}{l} 1 : \frac{4^n}{k} = 1 \\ 1 : \frac{n4^{n-1}}{k} = 1 \\ 1 : \text{values for } k \text{ and } n \\ 1 : \text{verifications} \end{array} \right.$

$$h(x) = \frac{x^4}{256} \Rightarrow h(0) = 0.$$

$$h'(x) = \frac{4x^3}{256} \Rightarrow h'(0) = 0 \text{ and } h'(x) > 0 \text{ for } 0 < x < 4.$$

6 2012 #4 (AB but suitable for BC) a, b – No Calc – Scoring Guidelines:

(a) $f'(x) = \frac{1}{2}(25 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{25 - x^2}}, \quad -5 < x < 5$

2 : $f'(x)$

(b) $f'(-3) = \frac{3}{\sqrt{25 - 9}} = \frac{3}{4}$

$$f(-3) = \sqrt{25 - 9} = 4$$

An equation for the tangent line is $y = 4 + \frac{3}{4}(x + 3)$.

2 : $\left\{ \begin{array}{l} 1 : f'(-3) \\ 1 : \text{answer} \end{array} \right.$