AP Calculus BC Saturday Study Session #1: Computing Antiderivatives & Integrals

(With special thanks to Lin McMullin)

In this prep session we look at the computation of antiderivatives and integrals that you may be asked to do on the AP Calculus BC exam. There are few applications here; they will be considered in several other student study sessions. However, you may be required to compute an antiderivative or integral as part of an application problem.

The antiderivatives and integrals that appear on the AP exams are probably a lot simpler than many you have done in class. The exam does not include "monster" integrands. They are testing that you know the basic idea and can apply the various techniques.

In this session, we will consider mainly multiple choice questions since nearly all of the questions on antiderivatives and integrals are of this type. Antiderivatives and integrals on the free-response sections are almost entirely in the context of the problems and do not stand alone like those on the multiple-choice. Often they are on the calculator allowed section and as a result, there is no need to solve the problem by hand.

Calc BC Students need to be able to do the following:

- Estimate integrals using Riemann Sums (LRAM, RRAM, MRAM, trapezoidal)
- Apply integration rules (sum/difference, constant multiplication)
- Integrate commonly used functions (power, trigonometric, exponential, logarithmic) – some of these may require simplification first
- Use u-substitution
- Use integration by parts
- Use partial fractions
- Integrate improper integrals (covered in a later session)
- Apply the FTC: $f(x) = f(a) + \int_a^x f'(t)dt$

Integration by Parts Refresher

$$\int u\,dv = uv - \int v\,du$$

Stumped about which part should be *u* and *dv*? Use the acronym DETAIL to help you to decide what *dv* should be! The higher the function appears on the list, the better it will work for *dv* in an integration by parts problem.

D – *dv* (just a friendly reminder that this is what it will help you to find)
E – Exponential functions such as e^x
T – Trigonometric functions such as $\sin x$ or $\cos x$
A – Algebraic functions such as x^2 or $6x^8$
I – Inverse trig functions such as $\tan^{-1} x$
L – Logarithmic functions such as $\ln x$ or $\log x$

Tabular Integration (a.k.a, Rapid Repeated Integration by Parts)

This is a nifty trick that can help you when a problem requires multiple uses of integration by parts. For example, consider the following problem:

$$\int x^4 \sin x \, dx$$

Using the DETAIL trick, we see that $dv = \sin x$ and so $u = x^4$. Set up your table as follows:

Derivatives of *u*		**Integrals of *dv***
x^4	+	$\sin x$
$4x^3$	−	$-\cos x$
$12x^2$	+	$-\sin x$
$24x$	−	$\cos x$
24	+	$\sin x$
0		$-\cos x$

Be sure to alternate the signs of the arrows in your table (always start with a plus sign) and stop when you've got a zero in the first column. Put it all together as follows:

$$\begin{aligned}\int x^4 \sin x \, dx &= x^4(-\cos x) - 4x^3(-\sin x) + 12x^2(\cos x) - 24x(\sin x) + 24(-\cos x) + C \\ &= -x^4 \cos x + 4x^3 \sin x + 12x^2 \cos x - 24x \sin x - 24 \cos x + C\end{aligned}$$

Multiple Choice Questions

Riemann Sums

1. **2003 #25 (BC) - No Calc:**

x	2	5	10	14
$f(x)$	12	28	34	30

The function f is continuous on the closed interval [2, 14] and has values as shown in the table above. Using the subintervals [2, 5], [5,10], and [10, 14], what is the approximation of $\int_2^{14} f(x)dx$ found by using a right Riemann sum?

a. 296　　c. 343　　e. 390

b. 312　　d. 374

2. **1998 #85 (BC) - Calc OK:**

x	2	5	7	8
$f(x)$	10	30	40	20

The function f is continuous on the closed interval [2, 8] and has values that are given in the table above. Using the subintervals [2, 5], [5, 7], and [7, 8], what is the trapezoidal approximation of $\int_2^{8} f(x)dx$?

a. 110　　c. 160　　e. 210

b. 130　　d. 190

3. **1998 #91 (BC) - Calc OK:**

t (sec)	0	2	4	6
$a(t)$ (ft/sec^2)	5	2	8	3

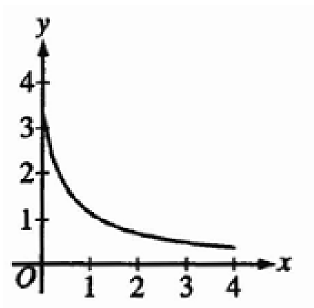
The data for the acceleration $a(t)$ of a car from 0 to 6 seconds are given in the table above. If the velocity at $t = 0$ is 11 feet per second, the approximate value of the velocity at $t = 6$, computed using a left-hand Riemann sum with three subintervals of equal length, is

a. 26 ft/sec　　c. 37 ft/sec　　e. 41 ft/sec

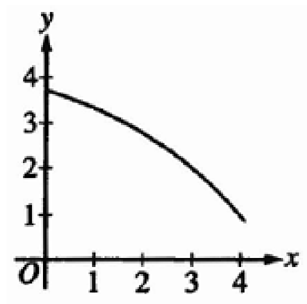
b. 30 ft/sec　　d. 39 ft/sec

4. **2003 #85 (BC) - Calc OK:** If a trapezoidal sum overapproximates $\int_0^4 f(x)dx$ and a right Riemann sum underapproximates $\int_0^4 f(x)dx$, which of the following could be the graph of $y = f(x)$?

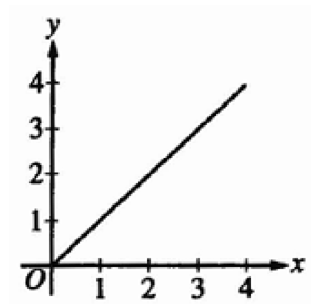
a.



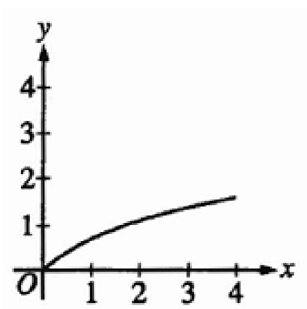
c.



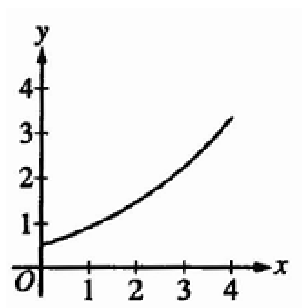
e.



b.



d.



Integration Rules

5. **1998 #11 (BC) - No Calc:** If f is a linear function and $0 < a < b$, then $\int_a^b f''(x)dx =$

a. 0

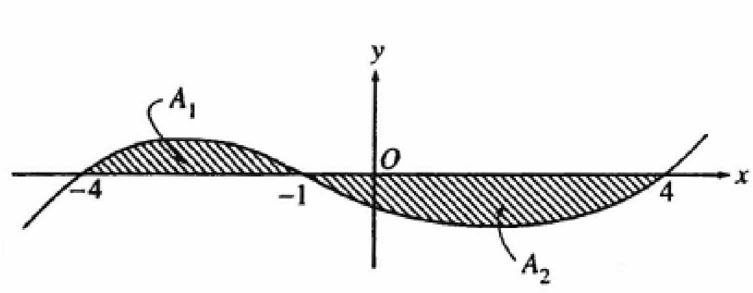
b. 1

c. $\frac{ab}{2}$

d. $b - a$

e. $\frac{b^2 - a^2}{2}$

6. **1997 #19 (BC) - No Calc:**



The graph of $y = f(x)$ is shown in the figure above. If A_1 and A_2 are positive numbers that represent the areas of the shaded regions, then in terms of A_1 and A_2, $\int_{-4}^{4} f(x)dx - 2\int_{-1}^{4} f(x)dx =$

a. A_1

b. $A_1 - A_2$

c. $2A_1 - A_2$

d. $A_1 + A_2$

e. $A_1 + 2A_2$

7. **1998 #82 (BC) - Calc OK:** If $f(x) = g(x) + 7$ for $3 \le x \le 5$, then $\int_3^5 \left[f(x) + g(x)\right]dx =$

a. $2\int_3^5 g(x)dx + 7$

b. $2\int_3^5 g(x)dx + 14$

c. $2\int_3^5 g(x)dx + 28$

d. $\int_3^5 g(x)dx + 7$

e. $\int_3^5 g(x)dx + 14$

Commonly Used Functions

8. **1998 #7 (BC) - No Calc:** $\int_1^e \left(\frac{x^2-1}{x}\right)dx =$

a. $e - \frac{1}{e}$

b. $e^2 - e$

c. $\frac{e^2}{2} - e + \frac{1}{2}$

d. $e^2 - 2$

e. $\frac{e^2}{2} - \frac{3}{2}$

9. **1997 #1 (BC) - No Calc:** $\int_0^1 \sqrt{x}\,(x+1)dx =$

a. 0

b. 1

c. $\frac{16}{15}$

d. $\frac{7}{5}$

e. 2

10. **1997 #82 (BC) - Calc OK:** If $0 \le x \le 4$, of the following, which is the greatest value of x such that $\int_0^x \left(t^2 - 2t\right)dt \ge \int_2^x t\,dt$?

a. 1.35

b. 1.38

c. 1.41

d. 1.48

e. 1.59

U-Substitution

11. **2003 #3 (BC) - No Calc:** $\int (3x+1)^5 dx =$

a. $\frac{(3x+1)^6}{18} + C$

b. $\frac{(3x+1)^6}{6} + C$

c. $\frac{(3x+1)^6}{2} + C$

d. $\frac{\left(\frac{3x^2}{2} + x\right)^6}{2} + C$

e. $\left(\frac{3x^2}{2} + x\right) + C$

12. **2003 #8 (BC) - No Calc:** $\int x^2 \cos\left(x^3\right)dx =$

a. $\frac{-1}{3}\sin\left(x^3\right) + C$

b. $\frac{1}{3}\sin\left(x^3\right) + C$

c. $\frac{-x^3}{3}\sin\left(x^3\right) + C$

d. $\frac{x^3}{3}\sin\left(x^3\right) + C$

e. $\frac{x^3}{3}\sin\left(\frac{x^4}{4}\right) + C$

13. **1998 #8 (BC) - No Calc:** If $\dfrac{dy}{dx} = \sin x \cos^2 x$ and if $y = 0$ when $x = \dfrac{\pi}{2}$, what is the value of y when $x = 0$?

a. -1

b. $\dfrac{-1}{3}$

c. 0

d. $\dfrac{1}{3}$

e. 1

Integration by Parts

14. **2003 #23 (BC) - No Calc:** $\int x \sin(6x)\,dx =$

a. $-x\cos(6x) + \sin(6x) + C$

b. $\dfrac{-x}{6}\cos(6x) + \dfrac{1}{36}\sin(6x) + C$

c. $\dfrac{-x}{6}\cos(6x) + \dfrac{1}{6}\sin(6x) + C$

d. $\dfrac{x}{6}\cos(6x) + \dfrac{1}{36}\sin(6x) + C$

e. $6x\cos(6x) - \sin(6x) + C$

15. **1997 #84 (BC) - Calc OK:** $\int x^2 \sin x\,dx =$

a. $-x^2\cos x - 2x\sin x - 2\cos x + C$

b. $-x^2\cos x + 2x\sin x - 2\cos x + C$

c. $-x^2\cos x + 2x\sin x + 2\cos x + C$

d. $\dfrac{-x^3}{3}\cos x + C$

e. $2x\cos x + C$

16. **1998 #15 (BC) - No Calc:** $\int x\cos x\,dx =$

a. $x\sin x - \cos x + C$

b. $x\sin x + \cos x + C$

c. $-x\sin x + \cos x + C$

d. $x\sin x + C$

e. $\dfrac{1}{2}x^2\sin x + C$

Partial Fractions

17. **2003 #26 (BC) - No Calc:** $\int \frac{2x}{(x+2)(x+1)}\,dx =$

a. $\ln|x+2| + \ln|x+1| + C$

b. $\ln|x+2| + \ln|x+1| - 3x + C$

c. $-4\ln|x+2| + 2\ln|x+1| + C$

d. $4\ln|x+2| - 2\ln|x+1| + C$

e. $2\ln|x| + \frac{2}{3}x + \frac{1}{2}x^2 + C$

18. **1998 #4 (BC) - No Calc:** $\int \frac{1}{x^2 - 6x + 8}\,dx =$

a. $\frac{1}{2}\ln\left|\frac{x-4}{x-2}\right| + C$

b. $\frac{1}{2}\ln\left|\frac{x-2}{x-4}\right| + C$

c. $\frac{1}{2}\ln|(x-2)(x-4)| + C$

d. $\frac{1}{2}\ln|(x-4)(x+2)| + C$

e. $\ln|(x-2)(x-4)| + C$

19. **1997 #86 (BC) - Calc OK:** $\int \frac{dx}{(x-1)(x+3)} =$

a. $\frac{1}{4}\ln\left|\frac{x-1}{x+3}\right| + C$

b. $\frac{1}{4}\ln\left|\frac{x+3}{x-1}\right| + C$

c. $\frac{1}{2}\ln|(x-1)(x+3)| + C$

d. $\frac{1}{2}\ln\left|\frac{2x+2}{(x-1)(x+3)}\right| + C$

e. $\ln|(x-1)(x+3)| + C$

FTC

20. **1997 #89 (BC) - Calc OK:** If f is the antiderivative of $\frac{x^2}{1+x^5}$ such that $f(1) = 0$, then $f(4) =$

a. -0.012

b. 0

c. 0.016

d. 0.376

e. 0.629

Solutions:

1. D
2. C
3. E
4. A
5. A
6. D
7. B
8. E
9. C
10. B
11. A
12. B
13. B
14. B
15. C
16. B
17. D
18. A
19. A
20. D

Free Response Questions

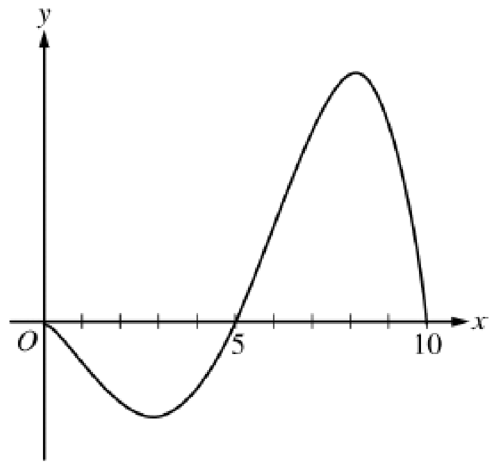
❶ 2012 #4 (BC) b – No Calc

x	1	1.1	1.2	1.3	1.4
$f'(x)$	8	10	12	13	14.5

The function f is twice differentiable for $x > 0$ with $f(1) = 15$ and $f''(1) = 20$. Values of f', the derivative of f, are given for selected values of x in the table above.

(b) Use a midpoint Riemann sum with two subintervals of equal length and values from the table to approximate $\int_1^{1.4} f'(x)\,dx$. Use the approximation for $\int_1^{1.4} f'(x)\,dx$ to estimate the value of $f(1.4)$. Show the computations that lead to your answer.

❷ 2011B #4 (BC) b – No Calc



Graph of f

The graph of the differentiable function $y = f(x)$ with domain $0 \le x \le 10$ is shown in the figure above. The area of the region enclosed between the graph of f and the x-axis for $0 \le x \le 5$ is 10, and the area of the region enclosed between the graph of f and the x-axis for $5 \le x \le 10$ is 27. The arc length for the portion of the graph of f between $x = 0$ and $x = 5$ is 11, and the arc length for the portion of the graph of f between $x = 5$ and $x = 10$ is 18. The function f has exactly two critical points that are located at $x = 3$ and $x = 8$.

(b) Evaluate $\int_0^{10} (3f(x) + 2)\, dx$. Show the computations that lead to your answer.

❸ 2010 #2 (BC) b – Calc OK

t (hours)	0	2	5	7	8
$E(t)$ (hundreds of entries)	0	4	13	21	23

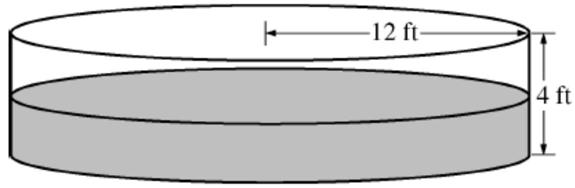
A zoo sponsored a one-day contest to name a new baby elephant. Zoo visitors deposited entries in a special box between noon $(t = 0)$ and 8 P.M. $(t = 8)$. The number of entries in the box t hours after noon is modeled by a differentiable function E for $0 \le t \le 8$. Values of $E(t)$, in hundreds of entries, at various times t are shown in the table above.

(b) Use a trapezoidal sum with the four subintervals given by the table to approximate the value of $\frac{1}{8}\int_0^8 E(t)\,dt$.

Using correct units, explain the meaning of $\frac{1}{8}\int_0^8 E(t)\,dt$ in terms of the number of entries.

❹ 2010B #3 (BC) a – Calc OK

t	0	2	4	6	8	10	12
$P(t)$	0	46	53	57	60	62	63



The figure above shows an aboveground swimming pool in the shape of a cylinder with a radius of 12 feet and a height of 4 feet. The pool contains 1000 cubic feet of water at time $t = 0$. During the time interval $0 \le t \le 12$ hours, water is pumped into the pool at the rate $P(t)$ cubic feet per hour. The table above gives values of $P(t)$ for selected values of t. During the same time interval, water is leaking from the pool at the rate $R(t)$ cubic feet per hour, where $R(t) = 25e^{-0.05t}$. (Note: The volume V of a cylinder with radius r and height h is given by $V = \pi r^2 h$.)

(a) Use a midpoint Riemann sum with three subintervals of equal length to approximate the total amount of water that was pumped into the pool during the time interval $0 \le t \le 12$ hours. Show the computations that lead to your answer.

❺ 2010 #1 (BC) a – Calc OK

There is no snow on Janet's driveway when snow begins to fall at midnight. From midnight to 9 A.M., snow accumulates on the driveway at a rate modeled by $f(t) = 7te^{\cos t}$ cubic feet per hour, where t is measured in hours since midnight. Janet starts removing snow at 6 A.M. $(t = 6)$. The rate $g(t)$, in cubic feet per hour, at which Janet removes snow from the driveway at time t hours after midnight is modeled by

$$g(t) = \begin{cases} 0 & \text{for } 0 \le t < 6 \\ 125 & \text{for } 6 \le t < 7 \\ 108 & \text{for } 7 \le t \le 9. \end{cases}$$

(a) How many cubic feet of snow have accumulated on the driveway by 6 A.M.?

❻ 2009 #5 (BC) b – No Calc

x	2	3	5	8	13
$f(x)$	1	4	–2	3	6

Let f be a function that is twice differentiable for all real numbers. The table above gives values of f for selected points in the closed interval $2 \le x \le 13$.

(b) Evaluate $\int_2^{13} (3 - 5f'(x))\, dx$. Show the work that leads to your answer.

❼ 2008 #2 (BC) d – Calc OK

t (hours)	0	1	3	4	7	8	9
$L(t)$ (people)	120	156	176	126	150	80	0

Concert tickets went on sale at noon $(t = 0)$ and were sold out within 9 hours. The number of people waiting in line to purchase tickets at time t is modeled by a twice-differentiable function L for $0 \le t \le 9$. Values of $L(t)$ at various times t are shown in the table above.

(d) The rate at which tickets were sold for $0 \le t \le 9$ is modeled by $r(t) = 550te^{-t/2}$ tickets per hour. Based on the model, how many tickets were sold by 3 P.M. $(t = 3)$, to the nearest whole number?

❶ 2012 #4 (BC) b – No Calc – Scoring Guidelines:

(b) $\int_1^{1.4} f'(x)\,dx \approx (0.2)(10) + (0.2)(13) = 4.6$

$f(1.4) = f(1) + \int_1^{1.4} f'(x)\,dx$

$f(1.4) \approx 15 + 4.6 = 19.6$

$3: \begin{cases} 1: \text{midpoint Riemann sum} \\ 1: \text{Fundamental Theorem of Calculus} \\ 1: \text{answer} \end{cases}$

❷ 2011B #4 (BC) b – No Calc – Scoring Guidelines:

(b) $\int_0^{10} (3f(x) + 2)\,dx = 3\left(\int_0^5 f(x)\,dx + \int_5^{10} f(x)\,dx\right) + 20$

$= 3(-10 + 27) + 20 = 71$

2 : answer

❸ 2010 #2 (BC) b – Calc OK – Scoring Guidelines:

(b) $\frac{1}{8}\int_0^8 E(t)\,dt \approx$

$\frac{1}{8}\left(2\cdot\frac{E(0)+E(2)}{2} + 3\cdot\frac{E(2)+E(5)}{2} + 2\cdot\frac{E(5)+E(7)}{2} + 1\cdot\frac{E(7)+E(8)}{2}\right)$

$= 10.687$ or 10.688

$\frac{1}{8}\int_0^8 E(t)\,dt$ is the average number of hundreds of entries in the box between noon and 8 P.M.

$3: \begin{cases} 1: \text{trapezoidal sum} \\ 1: \text{approximation} \\ 1: \text{meaning} \end{cases}$

❹ 2010B #3 (BC) a – Calc OK – Scoring Guidelines:

(a) $\int_0^{12} P(t)\,dt \approx 46\cdot 4 + 57\cdot 4 + 62\cdot 4 = 660 \text{ ft}^3$

$2: \begin{cases} 1: \text{midpoint sum} \\ 1: \text{answer} \end{cases}$

❺ 2010 #1 (BC) a – Calc OK – Scoring Guidelines:

(a) $\int_0^6 f(t)\,dt = 142.274$ or 142.275 cubic feet

$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$

❻ 2009 #5 (BC) b – No Calc – Scoring Guidelines:

(b) $\int_2^{13} (3 - 5f'(x))\,dx = \int_2^{13} 3\,dx - 5\int_2^{13} f'(x)\,dx$ $= 3(13-2) - 5(f(13) - f(2)) = 8$	2 : { 1 : uses Fundamental Theorem of Calculus 1 : answer

❼ 2008 #2 (BC) d – Calc OK – Scoring Guidelines:

(d) $\int_0^3 r(t)\,dt = 972.784$ There were approximately 973 tickets sold by 3 P.M.	2 : { 1 : integrand 1 : limits and answer