NATIONAL MATH + SCIENCE INITIATIVE

Taylor Series
(BC Only)

Taylor series is a very important topic on the BC Exam. You can expect to see several multiple choice questions and at least one free response question on the exam. It is necessary for you to understand both the mechanics and the philosophy behind this topic.

Philosophically, Taylor series provide a way to find a polynomial "look-alike" to a non-polynomial function. This is done by a specific formula shown below (which you should memorize)

Taylor Series centered at $x = 0$ (Maclaurin Series).
Let f be a function with derivatives of all orders on an interval containing $x = 0$. Then f, centered at $x = 0$, can be represented by

$$f(x) = \frac{f(0)}{0!}x^0 + \frac{f'(0)}{1!}x^1 + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

A Taylor series can be centered at any other location as well by the formula

Taylor Series centered at $x = a$
Let f be a function with derivatives of all orders on an interval containing $x = a$. Then f, centered at $x = a$, can be represented by

$$f(x) = \frac{f(a)}{0!}(x-a)^0 + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

Example: Let $f(x) = \ln(1+x)$. Find the first 4 terms of the Maclaurin series representation.

First, we have to find the values of f and its appropriate derivatives at $x = 0$.

$$f(x) = \ln(1+x)\Big|_{x=0} = 0$$
$$f'(x) = (1+x)^{-1}\Big|_{x=0} = 1$$
$$f''(x) = -1(1+x)^{-2}\Big|_{x=0} = -1$$
$$f'''(x) = 2(1+x)^{-3}\Big|_{x=0} = 2$$
$$f^{(iv)}(x) = -6(1+x)^{-4}\Big|_{x=0} = -6$$

Then, we insert these values into the Maclaurin formula giving

$$0 + \frac{1}{1!}x - \frac{1}{2!}x^2 + \frac{2}{3!}x^3 - \frac{6}{4!}x^4$$
$$\Rightarrow x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$$

Generally, it is not necessary to simplify your results on the Free Response section. Answers will be simplified on the Multiple Choice section.

Topic: Taylor Series and Error Bounds
Student Saturday Session

There are seven standard formulas or templates that you must know. You will need to know the function, the expansion, the sigma notation, and the interval of convergence. At least one of the items will be tested on the exam. Since you don't know which one will be tested, you must be ready for any and all of them.

The seven formulas are as follows:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots = \sum_{n=0}^{\infty} x^n; \quad -1 < x < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots + (-1)^n x^n + \ldots = \sum_{n=0}^{\infty} (-1)^n x^n; \quad -1 < x < 1$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad -\infty < x < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}; \quad -\infty < x < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}; \quad -\infty < x < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n-1} \frac{x^n}{n} + \ldots = \sum_{n=1}^{\theta} (-1)^{n-1} \frac{x^n}{n}; \quad -1 < x \le 1$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)}; \quad -1 \le x \le 1$$

There are three main questions that you can expect to be asked on the exam: 1) Write a function in terms of a series, 2) Find an error bound on an nth degree Taylor Polynomial, and 3) Find an interval of convergence. We will look at an example of each of these.

1. Writing a function in terms of a series. Find a Maclaurin series for the function $x\cos x$. To do this, look for a familiar function. We know a series expansion for $\cos x$ which is

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots$$

We just need to multiply the result by x giving

$$x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots\right) \quad \Rightarrow x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n)!} + \ldots$$

The ellipses (…) are required *before* and *after* the general term. Otherwise, you will lose a point.

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Error Bounds

To determine an error bound for a Taylor polynomial, you must first classify your polynomial as either an alternating or non-alternating series. Their error bounds are found as follows

Alternating Series
When a series is alternating, the error is maximized in the next unused term evaluated at the difference between the center of the convergence and the x-coordinate being evaluated.

Example: Let $f(x) = e^{-2x^2}$

A) Write the first four terms and the general term of the Taylor series expansion for *f*.

B) Let *g* be the function given by the sum of the first four nonzero terms of the power series for *f* about *x*= *0*. Show that $\left|f(x) - g(x)\right| < 0.02$ for $-0.6 \le x \le 0.6$.

Solution

A) Use the template to write the expansion for e^x.

$$e^x = 1 + x^1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

Next, substitute $-2x^2$ for *x* and simplify.

$$e^{-2x^2} = 1 + \left(-2x^2\right)^1 + \frac{\left(-2x^2\right)^2}{2!} + \frac{\left(-2x^2\right)^3}{3!} + \ldots + \frac{\left(-2x^2\right)^n}{n!} + \ldots$$

$$\Rightarrow 1 - 2x^2 + \frac{4x^4}{2!} - \frac{8x^6}{3!} + \ldots + (-1)^n \frac{2^n x^{2n}}{n!} + \ldots$$

B) Since we are using the first four terms of an alternating series, the error will be tied up in the fifth term evaluated at *x = 0.6*.

$$a_5 = \left|\frac{16x^8}{4!}\right|_{x=0.6} = \frac{16(0.6)^8}{24} = 0.11197 < 0.02$$

Non-Alternating Series
If a series is non-alternating, the error is still tied up in the next term by the formula

$Error < \left|\frac{f^{(n+1)}(z)}{(n+1)!}\left(x-a\right)^{n+1}\right|$ where $f^{(n+1)}(z)$ is the maximum value that the *(n+1)* derivative can take on the interval. We are not ever required to find *z*. We have to be told in some way what the maximum value for that derivative will be on the interval.

Example: Let *f* be a function with derivatives of all orders for $x > 0$.

Let $T_3(x) = 7 - 2(x-3) + 4(x-3)^2 + \frac{5}{3}(x-3)^3$ be the third degree Taylor polynomial for *f* centered at $x = 3$.

A) Use $T_3(x)$ to approximate *f(3.4)*

B) Suppose that $\left|f^{(4)}(x)\right| < 5$ for all values of x on [3, 4], find the maximum error using $T_3(3.4)$

Solution:

A) $f(3.4) \approx T(3.4) = 7 - 2(3.4-3) + 4(3.4-3)^2 + \frac{5}{3}(3.4-3)^3 = 6.947$

B)
$$Error < \left|\frac{f^{(4)}(z)}{4!}(x-3)^4\right|, \textit{ but } f^{(4)}(z) \le 5$$
$$\Rightarrow Error < \left|\frac{5}{24}(3.4-3)^4\right| = 0.0053$$

Interval of Convergence for Taylor Series

When looking for the interval of convergence for a Taylor Series, refer back to the interval of convergence for each of the basic Taylor Series formulas. Fit your function to the function being tested.

Example: Find the interval of convergence for $f(x) = \frac{x}{1+x}$.

Solution: $\frac{x}{1+x} = x\left(\frac{1}{1+x}\right)$. The basic series is $\frac{1}{1+x}$ which has an interval of convergence of $-1 < x < 1$. So, f has the same interval of convergence $-1 < x < 1$.

Sometimes, the exam will manipulate a Taylor series to a power series before asking for the interval of convergence. The most common test to find the interval of convergence for a power series is the Ratio Test, which says that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$. If $L < 1$, the series converges. If $L > 1$, the series diverges. If $L = 1$, the test fails and you must use another test. When using the Ratio Test, it is important to remember that the Ratio Test only checks the *open* interval. You must check the endpoints of the interval separately to determine if the interval is open or closed. The exam will never prompt you for this, you must remember it yourself.

Example: Find the interval of convergence for the series $\sum_{n=0}^{\infty} \frac{(x-2)^n}{2n}$.

First, apply the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(x-2)^{n+1}}{2(n+1)}}{\frac{(x-2)^n}{2n}}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{2(n+1)} \cdot \frac{2n}{(x-2)^n}\right|$$

$$= \lim_{n\to\infty}\left(\frac{2n}{2n+2}\right)|x-2| = |x-2|$$

This series will converge absolutely when $|x-2|<1 \to 1<x<3$.

Next, we check the series at each endpoint:

When $x = 1$, the series becomes $\sum_{n=1}^{\infty}\frac{(-1)^n}{2n}$, which converges conditionally by the Alternating Series Test.

When $x = 3$, the series becomes $\sum_{n=1}^{\infty}\frac{1}{2n}$, which diverges by the Limit Comparison Test with the series $\sum_{n=1}^{\infty}\frac{1}{n}$ (which diverges by the p-series test.) So the interval of convergence is $1 \le x < 3$.

Multiple Choice Examples

Example 1: 1997 BC 17

17. Let f be the function given by $f(x) = \ln(3-x)$. The third-degree Taylor polynomial for f about $x = 2$ is

(A) $-(x-2)+\frac{(x-2)^2}{2}-\frac{(x-2)^3}{3}$

(B) $-(x-2)-\frac{(x-2)^2}{2}-\frac{(x-2)^3}{3}$

(C) $(x-2)+(x-2)^2+(x-2)^3$

(D) $(x-2)+\frac{(x-2)^2}{2}+\frac{(x-2)^3}{3}$

(E) $(x-2)-\frac{(x-2)^2}{2}+\frac{(x-2)^3}{3}$

Example 2: 1998 BC 14

14. What is the approximation of the value of sin 1 obtained by using the fifth-degree Taylor polynomial about $x = 0$ for $\sin x$?

(A) $1 - \frac{1}{2} + \frac{1}{24}$

(B) $1 - \frac{1}{2} + \frac{1}{4}$

(C) $1 - \frac{1}{3} + \frac{1}{5}$

(D) $1 - \frac{1}{4} + \frac{1}{8}$

(E) $1 - \frac{1}{6} + \frac{1}{120}$

Example 3: 2003 BC 28

28. What is the coefficient of x^2 in the Taylor series for $\frac{1}{(1+x)^2}$ about $x = 0$?

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) 1 (D) 3 (E) 6

Example 4: 2003 BC 77 (Calculator allowed)

77. Let $P(x) = 3x^2 - 5x^3 + 7x^4 + 3x^5$ be the fifth-degree Taylor polynomial for the function f about $x = 0$. What is the value of $f'''(0)$?

(A) -30 (B) -15 (C) -5 (D) $-\frac{5}{6}$ (E) $-\frac{1}{6}$

Topic: Taylor Series and Error Bounds
Student Saturday Session

Free Response Examples

Example 1: 2008 BC 3 (calculator allowed)

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

Let h be a function having derivatives of all orders for $x > 0$. Selected values for h and its first four derivatives are indicated in the table above. The function h and these four derivatives are increasing on the interval $1 \le x \le 3$.

a) Write the first degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$. Is this approximation greater or less than $h(1.9)$? Explain your answer.

b) Write the third-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$.

c) Use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x = 2$ approximates $h(1.9)$ with an error less than 3×10^{-4}.

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Example 2: 2007 BC 6 (non-calculator)

Let f be the function given by $f(x) = e^{-x^2}$.

a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

b) Use your answer from part (a) to find $\lim\limits_{x \to 0} \dfrac{1 - x^2 - f(x)}{x^4}$.

c) Write the first four nonzero terms of the Taylor series for $\int_0^{1/2} e^{-t^2}\, dt$.

d) Explain why the estimate found in part (c) differs from the actual value of $\int_0^{1/2} e^{-t^2}\, dt$ by less than $\frac{1}{200}$.

Example 3: 2005 BC 6 (non-calculator)

Let f be a function with derivatives of all orders and for which $f(2) = 7$. When n is odd, the nth derivative of f at $x = 2$ is 0. When n is even and $n \geq 2$, the nth derivative at $x = 2$ is given by $f^{(n)}(2) = \frac{(n-1)!}{3^n}$.

a) Write the sixth-degree Taylor polynomial for f about $x = 2$.

b) In the Taylor series for f about $x = 2$, what is the coefficient of $(x-2)^{2n}$ for $n \geq 1$?

c) Find the interval of convergence of the Taylor series for f about $x = 2$. Show the work that leads to your answer.

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Example 4: 2006B BC6

6. The function f is defined by $f(x) = \dfrac{1}{1+x^3}$. The Maclaurin series for f is given by

$$1 - x^3 + x^6 - x^9 + \cdots + (-1)^n x^{3n} + \cdots,$$

which converges to $f(x)$ for $-1 < x < 1$.

(a) Find the first three nonzero terms and the general term for the Maclaurin series for $f'(x)$.

(b) Use your results from part (a) to find the sum of the infinite series $-\dfrac{3}{2^2} + \dfrac{6}{2^5} - \dfrac{9}{2^8} + \cdots + (-1)^n \dfrac{3n}{2^{3n-1}} + \cdots$.

(c) Find the first four nonzero terms and the general term for the Maclaurin series representing $\int_0^x f(t)\,dt$.

(d) Use the first three nonzero terms of the infinite series found in part (c) to approximate $\int_0^{1/2} f(t)\,dt$. What are the properties of the terms of the series representing $\int_0^{1/2} f(t)\,dt$ that guarantee that this approximation is within $\dfrac{1}{10,000}$ of the exact value of the integral?

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Example 5: 2007B BC6

6. Let f be the function given by $f(x) = 6e^{-x/3}$ for all x.

(a) Find the first four nonzero terms and the general term for the Taylor series for f about $x = 0$.

(b) Let g be the function given by $g(x) = \int_0^x f(t)\,dt$. Find the first four nonzero terms and the general term for the Taylor series for g about $x = 0$.

(c) The function h satisfies $h(x) = kf'(ax)$ for all x, where a and k are constants. The Taylor series for h about $x = 0$ is given by

$$h(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

Find the values of a and k.