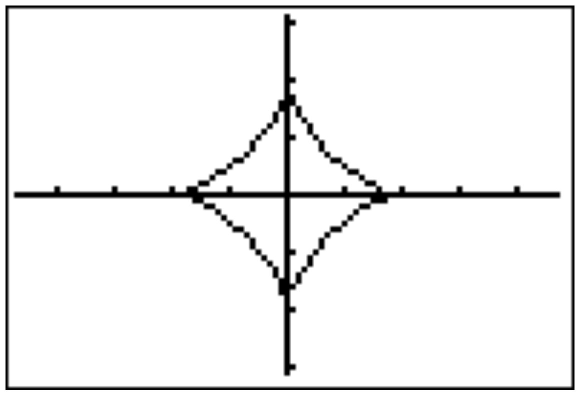
Parametric Functions and Vector Functions
(BC Only)

Parametric Functions

Parametric functions are another way of viewing functions. This time, the values of x and y are both dependent on another independent variable, usually t, for time. With parametric functions, we can draw very interesting curves such as the asteroid shown below.



We can use calculus to determine the slope of the curve at any point, to write an equation line tangent to the curve at a given point, and to find the length of a curve between two points.

The following formulas will be used on the AP Calculus exam. You need to know them.

The first derivative (the change in y with respect to x) is $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$.

The second derivative of y with respect to x is $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{dy'/dt}{dx/dt}$

The arc length is given by $L = \int_{t_1}^{t_2} \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dy}\right)^2}\, dt$.

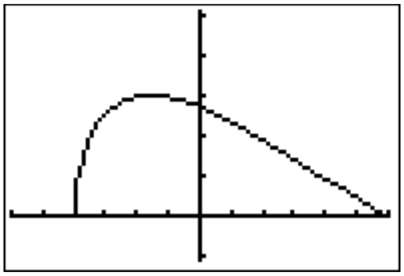
Example 1.

Consider the parametric function $\begin{cases} x = t^2 - 4 \\ y = 3\sin t \end{cases}$ $for\ 0 \le t \le \pi.$

a) Sketch the curve.
b) Find the highest point on the curve.
c) Find the length of the curve on the interval.

Solution

a)



b) To determine the highest point on the curve, we must maximize the *y*-coordinate.

$$\frac{d}{dt}(3\sin t) = 3\cos t.$$

$$3\cos t = 0 \Rightarrow t = \frac{\pi}{2}.$$

The sign of $3\cos t$ changes from positive to negative at $t = \frac{\pi}{2}$ which indicates a maximum value. Substituting $\frac{\pi}{2}$ into the original functions, we get (-1.533, 3).

c)

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dy}\right)^2}\, dt$$

$$\frac{dx}{dt} = 2t,\ \frac{dy}{dt} = 3\cos t$$

$$\Rightarrow L = \int_0^{\pi} \sqrt{(2t)^2 + (3\cos t)^2}\, dt$$

$$= 12.418$$

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Vector Functions

Vectors are quantities that have booth magnitude (size) and direction. They can be used to indicate motion in a two dimensional plane. We use the symbol $\langle a,b\rangle$ to represent a vector that stretches from the origin to the coordinates (a,b). Because vectors indicate motion, we can use calculus to measure the motion of the path traveled by the tip of the vector. The following relationships should be learned and memorized.

1. The particle's **position vector** is $\vec{r}(t)=\langle x(t),y(t)\rangle$.
2. The **magnitude** of the position vector is the length of the vector $L=\sqrt{(x(t))^2+(y(t))^2}$.
3. The **velocity vector** is $\vec{v}(t)=\left\langle \frac{dx}{dt},\frac{dy}{dt}\right\rangle$.
4. The **speed** is the magnitude of the velocity vector $\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$
5. The **acceleration vector** is $\vec{a}(t)=\left\langle \frac{d^2x}{dt^2},\frac{d^2y}{dt^2}\right\rangle$.
6. The **displacement** from t_1 to t_2 is given by integrating the velocity vector $\vec{v}(t)=\langle v_1(t),v_2(t)\rangle$ or $\left\langle \int_{t_1}^{t_2} v_1(t)\,dt, \int_{t_1}^{t_2} v_2(t)\,dt\right\rangle$.
7. The **total distance traveled** by the position vector is given by $\int_{t_1}^{t_2}\sqrt{(v_1(t))^2+(v_2(t))^2}\,dt$.

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I. Multiple Choice Examples

Example 1. 1998 BC #21 (non-calculator)

21. The length of the path described by the parametric equations $x = \frac{1}{3}t^3$ and $y = \frac{1}{2}t^2$, where $0 \le t \le 1$, is given by

(A) $\int_0^1 \sqrt{t^2+1}\, dt$

(B) $\int_0^1 \sqrt{t^2+t}\, dt$

(C) $\int_0^1 \sqrt{t^4+t^2}\, dt$

(D) $\frac{1}{2}\int_0^1 \sqrt{4+t^4}\, dt$

(E) $\frac{1}{6}\int_0^1 t^2\sqrt{4t^2+9}\, dt$

Example 2. 2003 BC Exam #84 (calculator active)

84. A particle moves in the xy-plane so that its position at any time t is given by $x(t) = t^2$ and $y(t) = \sin(4t)$. What is the speed of the particle when $t = 3$?

(A) 2.909 (B) 3.062 (C) 6.884 (D) 9.016 (E) 47.393

Example 3. 2008 BC Exam #1

1. At time $t \ge 0$, a particle moving in the xy-plane has velocity vector given by $v(t) = \langle t^2, 5t \rangle$. What is the acceleration vector of the particle at time $t = 3$?

(A) $\left\langle 9, \frac{45}{2} \right\rangle$ (B) $\langle 6, 5 \rangle$ (C) $\langle 2, 0 \rangle$ (D) $\sqrt{306}$ (E) $\sqrt{61}$

Example 4. 1998 BC Exam #10 (non-calculator)

10. A particle moves on a plane curve so that at any time $t > 0$ its x-coordinate is $t^3 - t$ and its y-coordinate is $(2t-1)^3$. The acceleration vector of the particle at $t = 1$ is

(A) $(0,1)$ (B) $(2,3)$ (C) $(2,6)$ (D) $(6,12)$ (E) $(6,24)$

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II. Free Response Examples

Example 1. 2006 Form B -- BC 2 (calculator allowed)

An object moving along a curve in the *xy*-plane is at position $\left(x(t),\ y(t)\right)$ at time *t*, where

$$\frac{dx}{dt} = \tan\left(e^{-t}\right) \quad \text{and} \quad \frac{dy}{dt} = \sec\left(e^{-t}\right) \quad \text{for } t \geq 0.$$

At time $t = 1$, the object is a position (2, 3).

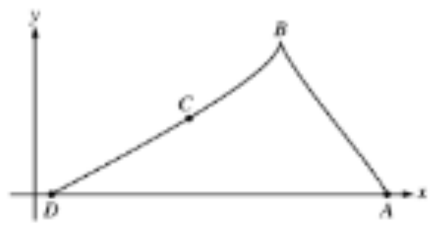
a) Write an equation for the line tangent to the curve at position (2, 3).

b) Find the acceleration vector and the speed of the object at time $t = 1$.

c) Find the total distance traveled by the object over the time interval $1 \leq t \leq 2$.

d) Is there a time $t \geq 0$ at which the object is on the *y*-axis? Explain why or why not.

Example 2. 2003 BC 2 (calculator allowed)



A particle starts at point A on the positive x-axis at time $t \geq 0$ and travels along the curve from A to B to C *to* D, as shown above. The coordinates of the particle's position $\left(x(t), y(t)\right)$ are differentiable functions of t, where $x'(t) = \dfrac{dx}{dt} = 9\cos\left(\dfrac{\pi t}{6}\right)\sin\left(\dfrac{\pi\sqrt{t+1}}{2}\right)$ and $y'(t) = \dfrac{dy}{dt}$ is not explicitly given. At time $t = 9$, the particle reaches its final position D on the positive x-axis.

a) At point C, is $\dfrac{dy}{dt}$ positive? At point C, is $\dfrac{dx}{dt}$ positive? Give a reason for each answer.

b) The slope of the curve is undefined at point B. At what time is the particle at point B?

c) The line tangent to the curve at the point $\left(x(8),\ y(8)\right)$ has equation $y=\frac{5}{9}x-2.$ Find the velocity vector and the speed of the particle at this point.

d) How far apart are points *A* and *D*, the initial and final positions, respectively, of the particle?

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Example 3. 2010 Form B – BC 2 (calculator allowed)

The velocity vector of a particle moving in the xy-plane has components given by

$$\frac{dx}{dt} = 14\cos\left(t^2\right)\sin\left(e^t\right) \quad \text{and} \quad \frac{dy}{dt} = 1 + 2\sin\left(t^2\right), \quad \text{for } 0 \le t \le 1.5.$$

At time $t = 0$, the position of the particle is (-2, 3).

a) For $0 \le t \le 1.5$, find all values of t at which the line tangent to the path of the particle is vertical.

b) Write an equation for the line tangent to the path of the particle at $t = 1$.

c) Find the speed of the particle at $t = 1$.

d) Find the acceleration vector of the particle at $t = 1$.

Example 4. 2011 BC 1 (calculator allowed)

At time t, a particle moving in the xy-plane is at position $(x(t),\ y(t))$ where $x(t)$ and $y(t)$ are not explicitly given. For $t \geq 0$, $\frac{dx}{dt} = 4t + 1$ and $\frac{dy}{dt} = \sin(t^2)$. At time $t = 0$, $x(0) = 0$ and $y(0) = -4$.

a) Find the speed of the particle at time $t = 3$, and find the acceleration vector of the particle at time $t = 3$.

b) Find the slope of the line tangent to the path of the particle at time $t = 3$.

c) Find the position of the particle at time $t = 3$.

d) Find the total distance traveled by the particle over the time interval $0 \leq t \leq 3$.