9.3 The Integral Test and *p* -series

two more ways to address whether a series converges or diverges

Remember: will NOT be able to answer what converges to (only Geometric and Telescoping do that)

Integral Test

Turn the series into a function
that must have these characteristics...

1. **positive**: if graphed,
   it's above the x axis
2. **continuous**: don't really show.
   use definition of polynomials
   or are continuous in their
   domain like e^x or ln x
3. **decreasing**: sometimes obvious,
   sometimes
   first derivative test.

Once you have these...
now set up an Improper Integral

$$\sum_{n=1}^{\infty} a_n$$

converges if $\int_1^{\infty}$ converges

diverges if $\int_1^{\infty}$ diverges

$\int$ value is **NOT** equal to the sum, it is only an indicator of convergence or divergence

Ex. 1 Use the Integral Test to determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{2}{3n+5}$$

pos. ✓
cont. ✓
dec. ✓

$\int_1^{\infty} \frac{2}{3x+5}\,dx$

$\lim_{b\to\infty} \int_1^{b} \frac{2}{3x+5}\,dx$

$\lim_{b\to\infty} 2\int_1^{b} \frac{1}{3x+5}\,dx$

$u = 3x+5$
$du = 3\,dx$
$\frac{1}{3}du = dx$

$\lim_{b\to\infty} \frac{2}{3}\int_1^{b} \frac{1}{u}\,du$

$\lim_{b\to\infty} \frac{2}{3} \ln|3x+5| \Big|_1^b$

$F(b) - F(1)$

$\lim_{b\to\infty} \frac{2}{3} \{\ln|3b+5| - \ln|8|\}$

∞

$\int$ diverges

$\therefore$ (series) Σ diverges by integral test

therefore

Ex. 2 Use the Integral Test to determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty}\left(\frac{\sin n}{n}\right)^2$$

pos.
cont.
~~dec.~~ oscillates
find another test

Ex. 3 Use the Integral Test to determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty}\frac{n}{n^4+1}$$

pos. ✓
cont. ✓
dec. ✓

(don't forget next two slides...good graphical representation)

$$\int_1^{\infty}\frac{x}{x^4+1}\,dx$$

$$\lim_{b\to\infty}\int_1^{b}\frac{x}{x^4+1}\,dx$$

$a^2=1$, $a=1$

$u^2=x^4$
$u=x^2$
$du=2x\,dx$
$\frac{1}{2}du=x\,dx$

$\int\frac{du}{a^2+u^2}=\frac{1}{a}\arctan\frac{u}{a}+C$

$$\lim_{b\to\infty}\frac{1}{2}\cdot\frac{1}{1}\arctan\frac{x^2}{1}\Big]_1^b$$

$$\lim_{b\to\infty}\frac{1}{2}\arctan x^2\Big]_1^b$$

$F(b)-F(1)$

$$\lim_{b\to\infty}\frac{1}{2}\left[\arctan b^2-\arctan 1\right]$$

$$\frac{1}{2}\left[\frac{\pi}{2}-\frac{\pi}{4}\right]$$

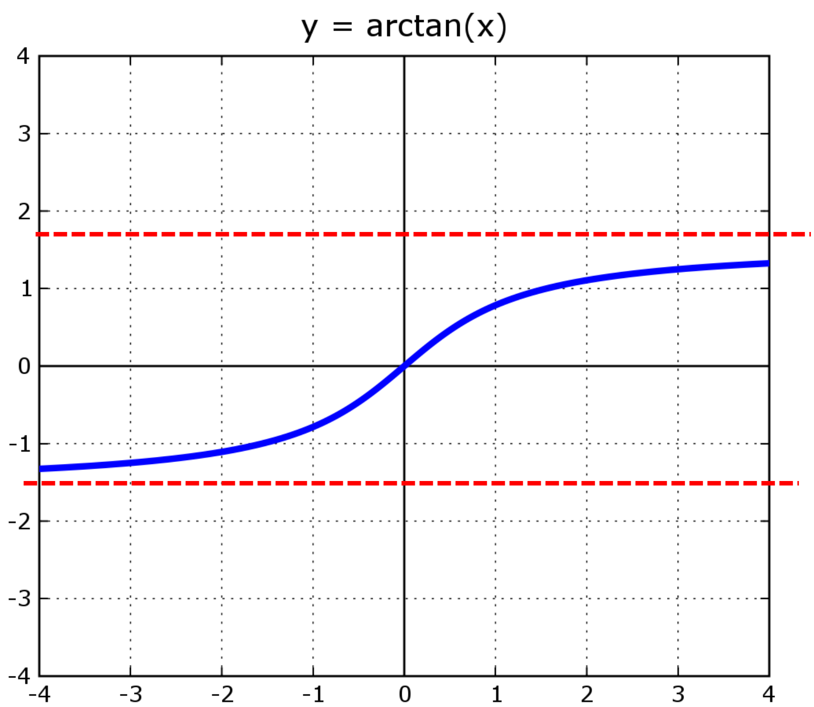
$$\frac{1}{2}\left[\frac{\pi}{4}\right]$$

$\frac{\pi}{8}$ converges

$\therefore$ Series conv. by integral test

Function	*Domain*	*Range*
arcsin $\sin^{-1}$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
arccos $\cos^{-1}$	$-1 \le x \le 1$	$0 \le y \le \pi$
arctan $\tan^{-1}$	$-\infty \le x \le \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

$$\frac{\pi}{2} \approx 1.57$$



Ex. 4

Use the Integral Test to determine whether the series converges or diverges.

pos. ✓
cont. ✓
dec. ✓

$$\sum_{n=1}^{\infty} ne^{-n^2} = \sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$$

$$\int_1^{\infty} xe^{-x^2}\,dx$$

$$\lim_{b\to\infty} \int_1^b xe^{-x^2}\,dx$$

$u = -x^2$
$du = -2x\,dx$
$-\frac{1}{2}du = x\,dx$

$$\lim_{b\to\infty} -\frac{1}{2}\int_1^b e^u\,du$$

$$\lim_{b\to\infty} -\frac{1}{2}\left[e^{-x^2}\right]_1^b$$

$F(b)-F(1)$

$$\lim_{b\to\infty} -\frac{1}{2}\left[e^{-b^2} - e^{-(1)^2}\right]$$

$$\lim_{b\to\infty} -\frac{1}{2}\left(\cancel{\frac{1}{e^{b^2}}} - \frac{1}{e}\right)$$

$\frac{1}{2e}$ conv.

∴ series conv. by integral test

Ex. 5

Use the Integral Test to determine whether the series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$$

pos. ✓
cont. ✓
dec.

$$y = \frac{\ln x}{x^{1/2}}$$

$$y' = \frac{x^{1/2}\cdot\frac{1}{x} - \ln x\left(\frac{1}{2}x^{-1/2}\right)}{(x^{1/2})^2}$$

$$y' = \frac{\frac{1}{x^{1/2}} - \frac{\ln x}{2x^{1/2}}}{x}$$

$$y' = \frac{\frac{2-\ln x}{2x^{1/2}}}{x}$$

$$y' = \frac{2-\ln x}{2x^{3/2}}$$

$2-\ln x = 0$ $\quad 2x^{3/2}=0$
$\ln x = 2$ $\quad x = 0$
$e^2 = x$
<9

dec.

$0 \quad e^2$ −

$\frac{-}{+}$

dec ✓

$$\int_1^{\infty} \frac{\ln x}{\sqrt{x}}\,dx$$

$$\lim_{b\to\infty} \int_1^b \ln x \cdot x^{-1/2}\,dx$$

L $\ln x$
I
A $x^{-1/2}\,dx$
T
E

$u = \ln x$ $\quad v = 2x^{\frac{1}{2}}$
$du = \frac{1}{x}dx$ $\quad dv = x^{-1/2}\,dx$

$uv - \int v\,du$

$$\lim_{b\to\infty}\left[\ln x\,(2x^{1/2}) - \int 2x^{1/2}\cdot\frac{1}{x}\,dx\right]$$

$$\lim_{b\to\infty}\left[\ln x\,(2x^{1/2}) - 2\int \frac{1}{x^{1/2}}\,dx\right]_2^{\infty}$$

$\downarrow$
∞
div.

∴ series div. by integral test