Warm up

$c=1$

Find a 4th degree Taylor Polynomial, centered at 1 for

$f(x) = \ln x$ $\quad f(1) = 0$

$f'(x) = \frac{1}{x} = x^{-1}$ $\quad f'(1) = 1$

$f''(x) = -x^{-2}$ $\quad f''(1) = -1$

$f'''(x) = 2x^{-3}$ $\quad f'''(1) = 2$

$f^{IV}(x) = -3\cdot2\cdot x^{-4}$ $\quad f^{IV}(1) = -3\cdot2$

$$P_{(n)}(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!} + \ldots \frac{f^n(c)(x-c)^n}{n!}$$

$$P_{(4)}(x) = 0 + 1(x-1) - \frac{1(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} - \frac{3\cdot2(x-1)^4}{4!}$$

$$P_{(4)}(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5}$$

Use $P_{(4)}(x)$ to estimate $f(1.1)$

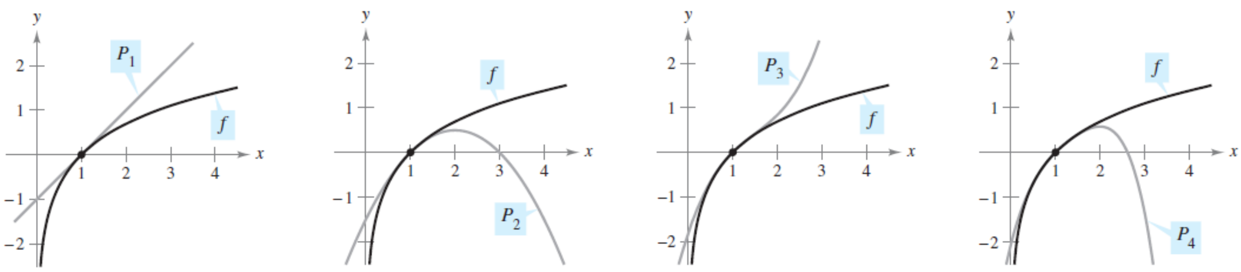
$$P_{(4)}(1.1) = .1 - \frac{(.1)^2}{2} + \frac{(.1)^3}{3} - \frac{(.1)^4}{4}$$

$$f(1.1) \approx P_{(4)}(1.1)$$

Error: because it's alternating at most the 1st omitted term

$$\text{error} \le \frac{(.1)^5}{5}$$

The more we expand the degree, the better the approximation of the polynomial.



9.10
Taylor and Maclaurin **Series**

What if it does not look like a geometric sum, like section 9.9?

(p. 676)

The Form of a Convergent Power Series

THEOREM 9.22 THE FORM OF A CONVERGENT POWER SERIES

If f is represented by a power series $f(x) = \Sigma\, a_n(x - c)^n$ for all x in an open interval I containing c, then $a_n = f^{(n)}(c)/n!$ and

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdots.$$

emphasize ellipses, addition sign and ellipses

Have we seen this before?

What's the power in a power series?

$\boxed{\text{Ability to derive new series!!!}}$

need to know p. 682...elementary functions...makes life way easier
could do long way like in 9.7, but worth it to memorize

POWER SERIES FOR ELEMENTARY FUNCTIONS

Function	*Interval of Convergence*
$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \cdots + (-1)^n(x-1)^n + \cdots$	$0 < x < 2$
$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots$	$-1 < x < 1$
$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + \frac{(-1)^{n-1}(x-1)^n}{n} + \cdots$	$0 < x \le 2$
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots$	$-\infty < x < \infty$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$	$-\infty < x < \infty$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$	$-\infty < x < \infty$
$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$	$-1 \le x \le 1$
$\arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots + \frac{(2n)!x^{2n+1}}{(2^n n!)^2(2n+1)} + \cdots$	$-1 \le x \le 1$
$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \frac{k(k-1)(k-2)(k-3)x^4}{4!} + \cdots$	$-1 < x < 1$*

* *The convergence at $x = \pm 1$ depends on the value of k.*

Ex. 1

Write a power series for $f(x) = e^{-3x}$

using $e^x = \sum_{n=0}^{\infty} \frac{(x)^n}{n!} = 1 + (x) + \frac{(x)^2}{2!} + \ldots + \frac{(x)^n}{n!} + \ldots$

$$e^{-3x} = \sum_{n=0}^{\infty} \frac{(-3x)^n}{n!}$$

$$\overset{n=0}{1} + \overset{n=1}{(-3x)} + \overset{n=2}{\frac{(-3x)^2}{2!}} + \ldots$$

$$e^{-3x} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n x^n}{n!}$$

Ex. 2

Write a power series for $f(x) = \sin(5x)$

using $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+1}}{(2n+1)!} = (x) - \frac{(x)^3}{3!} + \frac{(x)^5}{5!} + \ldots$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (5x)^{2n+1}}{(2n+1)!} = 5x - \frac{(5x)^3}{3!} + \frac{(5x)^5}{5!} + \ldots$$

Ex. 3

Write a power series for $h(x) = x\cos x$

using $x\cos x = x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) = x\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

$$x\cos x = x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!}$$

end of day 1

need to know p. 682...elementary functions...makes life way easier
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POWER SERIES FOR ELEMENTARY FUNCTIONS

Function	Interval of Convergence
$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \cdots + (-1)^n (x-1)^n + \cdots$	$0 < x < 2$
$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots$	$-1 < x < 1$
$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + \frac{(-1)^{n-1}(x-1)^n}{n} + \cdots$	$0 < x \le 2$
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots$	$-\infty < x < \infty$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$	$-\infty < x < \infty$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$	$-\infty < x < \infty$
$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$	$-1 \le x \le 1$
$\arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots + \frac{(2n)!x^{2n+1}}{(2^n n!)^2(2n+1)} + \cdots$	$-1 \le x \le 1$
$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \frac{k(k-1)(k-2)(k-3)x^4}{4!} + \cdots$	$-1 < x < 1$*

** The convergence at $x = \pm 1$ depends on the value of k.*

$$\int e^{t^2}\, dt$$

Ex. 4

$$\int_0^x e^{t^2}\,dt$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + (x) + \frac{(x)^2}{2!} + \ldots$$

$$\int e^{t^2}\,dt = \int \sum_{n=0}^{\infty} \frac{t^{2n}}{n!}\,dt = \int 1 + t^2 + \frac{t^4}{2!} + \cdots\,dt$$

$$\int_0^x \sum_{n=0}^{\infty} \frac{t^{2n}}{n!}\,dt \qquad \int_0^x 1 + t^2 + \frac{t^4}{2!} + \cdots\,dt$$

$$\left[\sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)n!}\right]_0^x \qquad \left. t + \frac{t^3}{3} + \frac{t^5}{5\cdot 2!} + \cdots \right]_0^x$$

$$F(x) - F(0) \qquad F(x) - F(0)$$

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!} - 0 \qquad \left(x + \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} + \cdots\right) - (0)$$

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!}$$

$n=0 \quad n=1 \quad n=2$

$$\frac{x^{0+1}}{1\,0!} + \frac{x^3}{3\cdot 1!} + \frac{x^5}{5\cdot 2!}$$

$$x + \frac{x^3}{3} + \frac{x^5}{5\cdot 2!}$$

Ex. 5

Find the first three terms and the **general term** for

$$\int_0^x \left(\frac{\sin t}{t}\right) dt$$

$$\frac{\sin t}{t} = \frac{\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}}{t} = \frac{t - \frac{t^3}{3!} + \frac{t^5}{5!} + \ldots}{t}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!} \qquad = \frac{t}{t} - \frac{\frac{t^3}{3!}}{t} + \frac{\frac{t^5}{5!}}{t}$$

$$\int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!}\,dt \qquad = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} + \cdots$$

$$\int_0^x 1 - \frac{t^2}{3!} + \frac{t^4}{5!} + \cdots\,dt$$

$$\left.\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)(2n+1)!}\right]_0^x \qquad \left. t - \frac{t^3}{3\cdot 3!} + \frac{t^5}{5\cdot 5} + \cdots \right]_0^x$$

$$F(x) - F(0) \qquad F(x) - F(0)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!} - 0 \qquad x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} + \cdots$$

$n=0 \quad n=1 \quad n=2$

$$\frac{x}{\;} - \frac{1\cdot x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} + \cdots$$