9.4 Comparison of Series

- Direct Comparison Test
- Limit Comparison Test

Direct Comparison Test

read top half of p. 626, (next slide)
what is the problem, and
how will we solve it?

Direct Comparison Test

For the convergence tests developed so far, the terms of the series have to be fairly simple and the series must have special characteristics in order for the convergence tests to be applied. A slight deviation from these special characteristics can make a test nonapplicable. For example, in the following pairs, the second series cannot be tested by the same convergence test as the first series even though it is similar to the first.

1. $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is geometric, but $\sum_{n=0}^{\infty} \frac{n}{2^n}$ is not.

2. $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a p-series, but $\sum_{n=1}^{\infty} \frac{1}{n^3 + 1}$ is not.

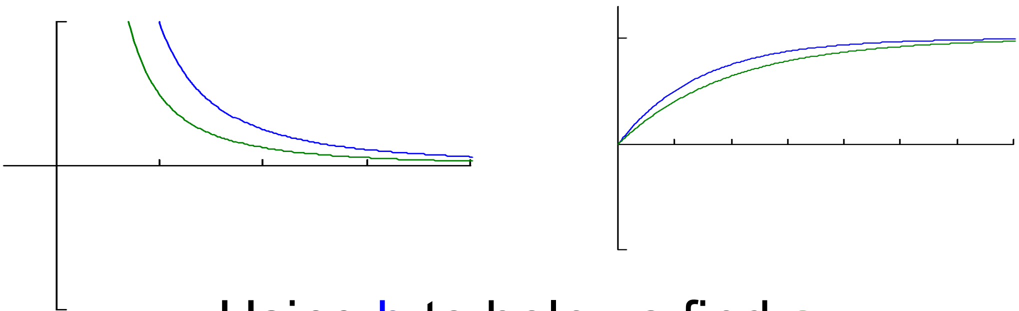
3. $a_n = \frac{n}{(n^2 + 3)^2}$ is easily integrated, but $b_n = \frac{n^2}{(n^2 + 3)^2}$ is not.

In this section you will study two additional tests for positive-term series. These two tests greatly expand the variety of series you are able to test for convergence or divergence. They allow you to *compare* a series having complicated terms with a simpler series whose convergence or divergence is known.

Direct Comparison Test

$0 < a \leq b$ (we know)

if b converges, then a converges



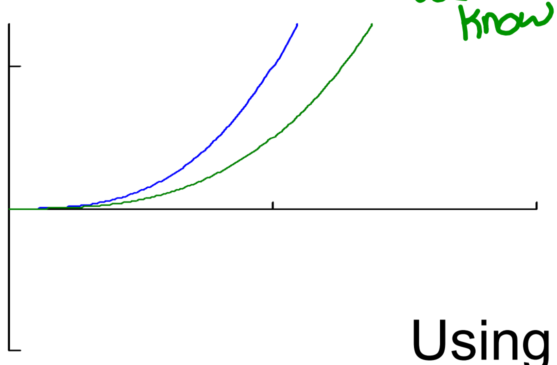
Using b to help us find a.

Direct Comparison Test

$0 < a \le b$

if a diverges, then b diverges

we know



Using a to help us find b.

Read the bottom of p. 627 below the blue line...
write in your own words for clarification

Remember that both parts of the Direct Comparison Test require that $0 < a_n \le b_n$. Informally, the test says the following about the two series with nonnegative terms.

1. If the "larger" series converges, the "smaller" series must also converge.
2. If the "smaller" series diverges, the "larger" series must also diverge.

Ex. 1

$$\sum_{n=1}^{\infty} \frac{1}{4^n + 5}$$

choose a twin

compare and label

$\frac{1}{4^n} = \left(\frac{1}{4}\right)^n$ conv. by geo.

$\frac{1}{4^n+5} \leq \frac{1}{4^n}$

unknown $\leq$ known

+ known conv.

then behave same way

conv. DCT to $\frac{1}{4^n}$

Ex. 2

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n} - 1}$$

choose a twin

compare and label

$\frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$ div. by p-series

$\frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{n} - 1}$

know

div.

and is less than

the unknown

behave same way

DCT to $\frac{1}{n^{1/2}}$ div.

Ex. 3

$$\sum_{n=1}^{\infty} \frac{1}{n!}$$

choose a twin

compare and label

$\frac{1}{n^2}$ conv. by p-series

$$\frac{1}{n!} \le \frac{1}{n^2}$$

$$\frac{1}{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} \le \frac{1}{10^2} \qquad n=10$$

DCT to $\frac{1}{n^2}$ conv.

Limit Comparison Test

if $a_n > 0$ and $b_n > 0$

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L$$

finite,positive number

Then both a_n and b_n ... converge or diverge

behave the same

In other words...

$$\lim_{n\to\infty} \frac{new}{known} \quad \text{if } = L$$

then matching behavior

**good for messy polynomials

Study table at top of p. 627-
how do we choose a comparison series?

Given Series	*Comparison Series*	*Conclusion*
$\sum_{n=1}^{\infty} \frac{1}{3n^2 - 4n + 5}$	$\sum_{n=1}^{\infty} \frac{1}{n^2}$	Both series converge.
$\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n - 2}}$	$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$	Both series diverge.
$\sum_{n=1}^{\infty} \frac{n^2 - 10}{4n^5 + n^3}$	$\sum_{n=1}^{\infty} \frac{n^2}{n^5} = \sum_{n=1}^{\infty} \frac{1}{n^3}$	Both series converge.

In other words, when choosing a series for comparison, you can disregard all but the *highest powers of n* in both the numerator and the denominator.

Ex. 1

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

$\sum \frac{n}{n^2} = \sum \frac{1}{n}$ div. by p-series

$$\lim_{n\to\infty} \frac{\frac{n}{n^2+1}}{\frac{1}{n}}$$

$$\lim_{n\to\infty} \frac{n}{n^2+1} \cdot \frac{n}{1}$$

1

LCT to $\sum_{n=1}^{\infty} \frac{1}{n}$

series Div.

Ex. 2

$$\sum_{n=1}^{\infty} \frac{5n-3}{n^3-2n+5}$$

$$\sum \frac{n}{n^3} = \sum \frac{1}{n^2}$$

conv. by p-series

$$\lim_{n\to\infty} \frac{\frac{5n-3}{n^3-2n+5}}{\frac{1}{n^2}}$$

$$\lim_{n\to\infty} \frac{5n-3}{n^3-2n+5} \cdot \frac{n^2}{1}$$

5 behave same

LCT to $\sum \frac{1}{n^2}$

series conv.

Ex. 3

$$\sum_{n=1}^{\infty} \frac{2\sqrt{n}}{n(n^2-3)}$$

$$\sum \frac{n^{1/2}}{n^3} = \sum \frac{1}{n^{5/2}}$$

conv. by p-series

$$\lim_{n\to\infty} \frac{\frac{2n^{1/2}}{n^3-3n}}{\frac{1}{n^{5/2}}}$$

$$\lim_{n\to\infty} \frac{2n^{1/2}}{n^3-3n} \cdot \frac{n^{5/2}}{1}$$

2

LCT to $\sum \frac{1}{n^{5/2}}$

series conv.

In summary...

limit comparison used a lot more often then direct comparison

direct comparison can come in handy, but there are other tests out there too

3 more to go