6.3 Separable Differential Equations

Ex. 2

Solve the differential equation:

$$\frac{dy}{dx} = x \cos x^2$$

$$\int dy = \int x \cos x^2 \, dx$$

$u = x^2$

$du = 2x\,dx$

$\frac{1}{2} du = x\,dx$

$$y = \frac{1}{2} \int \cos u \, du$$

$$\frac{1}{2} \sin x^2 + C$$

Ex. 3

Solve the differential equation:

$$\frac{dy}{dx} = x\sqrt{5-x}$$

$x = 5-u$ $\quad u = 5-x$

$du = -dx$

$-du = dx$

$$\int dy = \int x(5-x)^{1/2}\,dx$$

$$y = -\int x\, u^{1/2}\,du$$

$$y = -\int (5-u)u^{1/2}\,du$$

$$y = -\int 5u^{1/2} - u^{3/2}\,du$$

$$-\left[\frac{2}{3}\cdot 5u^{3/2} - \frac{2}{5}u^{5/2}\right] + C$$

$$y = -\frac{10}{3}(5-x)^{3/2} + \frac{2}{5}(5-x)^{5/2} + C$$

Ex. 4

Solve the differential equation:

$$\frac{dy}{dx} = \frac{x^2+2}{3y^2}$$

$$\int y^2\,dy = \frac{1}{3}\int (x^2+2)\,dx$$

$$\frac{y^3}{3} = \frac{1}{3}\left(\frac{x^3}{3} + 2x\right) + C$$

$$y^3 = \frac{x^3}{3} + 2x + C$$

Ex. 5

Solve the differential equation:

$$xy' = y$$

(remember: differentials always in numerator)

$$x\frac{dy}{dx} = y$$

$$x\,dy = y\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$

$$\ln|y| = \ln|x| + c$$

$$e^{\ln|y|} = e^{\ln|x| + c}$$

$$y = e^{\ln|x|} e^{c}$$

$$y = xC$$

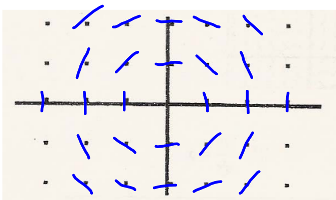
$$y = Cx$$

Ex. 6

a. Sketch the slope field, then solve by hand, .
b. Solve analytically (by hand)

see a lot!!

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{x}{y}$$



$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\int y\,dy = -\int x\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + c$$

$$y^2 = -x^2 + C$$

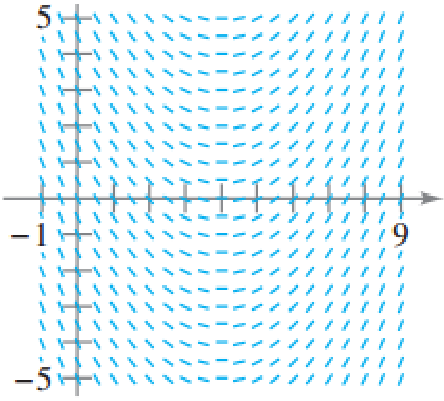
$$x^2 + y^2 = c$$

Ex. 7

Slope Fields **In Exercises 59–62, (a) write a differential equation for the statement, (b) match the differential equation with a possible slope field, and (c) verify your result by using a graphing utility to graph a slope field for the differential equation. [The slope fields are labeled (a), (b), (c), and (d).] To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

The rate of change of y with respect to x is proportional to the difference between x and 4.

$$\frac{dy}{dx} = K(x-4)$$



Stand and Deliver

Logistic Curves

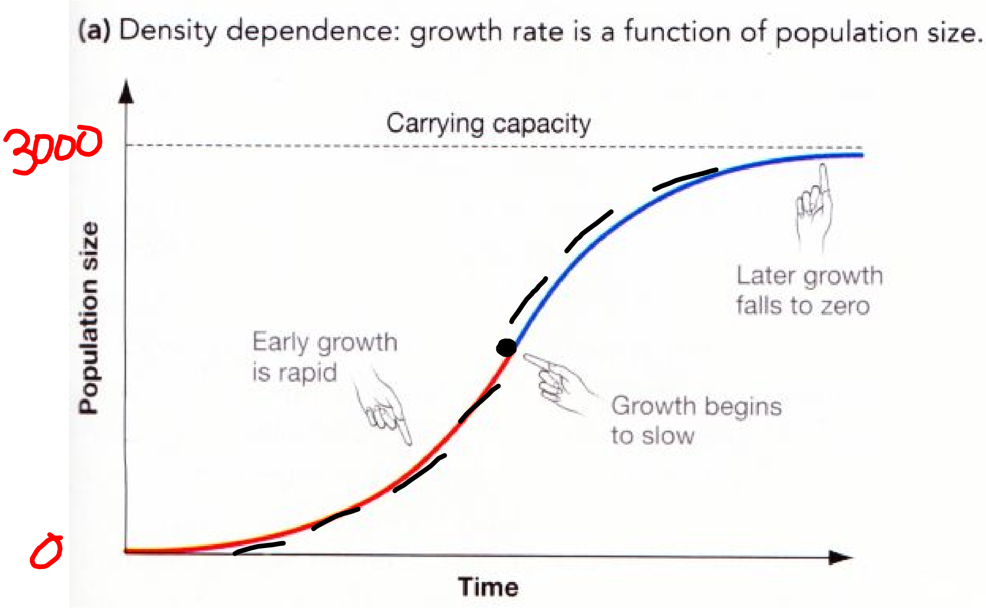
6.3

$$\boxed{\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)}$$

$$y = \frac{L}{1 + be^{-kt}}$$

L = Carry capacity (upper horizontal limit)
k = proportionality constant

memorize these two so avoid integrating on p. 427


(a) Density dependence: growth rate is a function of population size.
Carrying capacity
3000
Population size
0
Early growth
is rapid
Growth begins
to slow
Later growth
falls to zero
Time

EXAMPLE 9 Deriving the General Solution

Solve the logistic differential equation $\dfrac{dy}{dt} = ky\left(1 - \dfrac{y}{L}\right)$.

Solution Begin by separating variables.

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right) \qquad \text{Write differential equation.}$$

$$\frac{1}{y(1 - y/L)}dy = kdt \qquad \text{Separate variables.}$$

$$\int \frac{1}{y(1 - y/L)}dy = \int kdt \qquad \text{Integrate each side.}$$

$$\int \left(\frac{1}{y} + \frac{1}{L - y}\right)dy = \int kdt \qquad \text{Rewrite left side using partial fractions.}$$

$$\ln|y| - \ln|L - y| = kt + C \qquad \text{Find antiderivative of each side.}$$

$$\ln\left|\frac{L - y}{y}\right| = -kt - C \qquad \text{Multiply each side by } -1 \text{ and simplify.}$$

$$\left|\frac{L - y}{y}\right| = e^{-kt-C} = e^{-C}e^{-kt} \qquad \text{Exponentiate each side.}$$

$$\frac{L - y}{y} = be^{-kt} \qquad \text{Let } \pm e^{-C} = b.$$

Solving this equation for y produces $y = \dfrac{L}{1 + be^{-kt}}$. ■

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26. The population $P(t)$ of a species satisfies the logistic differential equation $\dfrac{dP}{dt} = P\left(2 - \dfrac{P}{5000}\right)$, where the initial population $P(0) = 3,000$ and t is the time in years. What is $\lim\limits_{t\to\infty} P(t)$?

(A) 2,500 (B) 3,000 (C) 4,200 (D) 5,000 (E) 10,000

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$$

$$= 2P\left(1 - \frac{P}{10,000}\right)$$

Horizontal asymptote
Carrying capacity

answer...E

13. A population of wolves is modeled by the function P and grows according to the logistic differential equation $\frac{dP}{dt} = 5P\left(1 - \frac{P}{5000}\right)$, where t is the time in years and $P(0) = 1000$. Which of the following statements are true?

I. $\lim_{t \to \infty} P(t) = 5000$

II. $\frac{dP}{dt}$ is positive for $t > 0$.

III. $\frac{d^2P}{dt^2}$ is positive for $t > 0$.

(A) I only

(B) II only

(C) I and II only

(D) I and III only

(E) I, II, and III

ar

Ex. 1

Solve the differential equation:

$$\cancel{dy} \quad \frac{dy}{\cancel{dx}} = (x^3 - 4x)\,dx$$

$$\int dy = \int x^3 - 4x \, dx$$

$$y = \frac{x^4}{4} - \frac{4x^2}{2} + c$$

$$y = \frac{x^4}{4} - 2x^2 + c$$