## 2013 AP® CALCULUS BC FREE-RESPONSE QUESTIONS

5. Consider the differential equation $\frac{dy}{dx} = y^2(2x+2)$. Let $y = f(x)$ be the particular solution to the differential equation with initial condition $f(0) = -1$.

(b) Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f\left(\frac{1}{2}\right)$.

(c) Find $y = f(x)$, the particular solution to the differential equation with initial condition $f(0) = -1$.

## AP® CALCULUS BC
## 2013 SCORING GUIDELINES

**Question 5**

(b) $f\left(\frac{1}{4}\right) \approx f(0) + f'(0)\left(\frac{1}{4}\right)$

$= -1 + (2)\left(\frac{1}{4}\right) = -\frac{1}{2}$

$f\left(\frac{1}{2}\right) \approx f\left(\frac{1}{4}\right) + f'\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)$

$= -\frac{1}{2} + \left(-\frac{1}{2}\right)^2\left(2 \cdot \frac{1}{4} + 2\right)\left(\frac{1}{4}\right) = -\frac{11}{32}$

$2 : \begin{cases} 1 : \text{Euler's method} \\ 1 : \text{answer} \end{cases}$

(c) $\frac{dy}{dx} = y^2(2x+2)$

$\frac{dy}{y^2} = (2x+2)\,dx$

$\int \frac{dy}{y^2} = \int (2x+2)\,dx$

$-\frac{1}{y} = x^2 + 2x + C$

$-\frac{1}{-1} = 0^2 + 2 \cdot 0 + C \Rightarrow C = 1$

$-\frac{1}{y} = x^2 + 2x + 1$

$y = -\frac{1}{x^2 + 2x + 1} = -\frac{1}{(x+1)^2}$

$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

$\Delta x \} = \frac{1}{4} = .25$

b)

| | | | |
|---|---|---|---|
| $x$ | $0$ | $\frac{1}{4}$ | $\frac{1}{2}$ |
| $y$ | $-1 \rightarrow$ | $-1+\frac{1}{2} = -\frac{1}{2} \rightarrow$ | $-\frac{1}{2}+\frac{5}{32} = \frac{-16}{32}+\frac{5}{32} = \frac{-11}{32}$ |
| $y^2(2x+2) = \frac{dy}{dx}$ | $(-1)^2(2(0)+2)$ <br> $1(2)$ <br> $2$ | $(-\frac{1}{2})^2(2(\frac{1}{4})+2)$ <br> $\frac{1}{4}(\frac{1}{2}+2)$ <br> $\frac{1}{4}(\frac{5}{2}) = \frac{5}{8}$ | |
| $dy = \underset{(\frac{1}{4})}{dx} \cdot \frac{dy}{dx}$ | $\frac{1}{4}(2)$ <br> $\frac{1}{2}$ | $\frac{1}{4}(\frac{5}{8})$ <br> $\frac{5}{32}$ | |

$2\frac{1}{2} = \frac{5}{2}$

$f(0) = -1$

c) $\frac{dy}{dx} = y^2(2x+2)$

$dy = y^2(2x+2)\,dx$

$\int \frac{1}{y^2}\,dy = \int (2x+2)\,dx$

$\int y^{-2}\,dy = \int (2x+2)\,dx$

$\frac{y^{-1}}{-1} = x^2+2x+C$

$-\frac{1}{y} = x^2+2x+C$

$-\frac{1}{1} = 0+0+C$

$1 = C$

$\frac{-1}{y} = \frac{x^2+2x+1}{1}$

$y(x^2+2x+1) = -1$

$y = \frac{-1}{x^2+2x+1}$

71.

***Forestry*** The value of a tract of timber is $V(t) = 100{,}000e^{0.8\sqrt{t}}$, where $t$ is the time in years, with $t = 0$ corresponding to 2008. If money earns interest continuously at 10%, the present value of the timber at any time $t$ is $A(t) = V(t)e^{-0.10t}$. Find the year in which the timber should be harvested to maximize the present value function.

$$A(t) = 100{,}000e^{.8t^{1/2}} \cdot e^{-.10t}$$

$$A(t) = 100{,}000e^{.8t^{1/2} - .10t}$$

$$A'(t) = 100{,}000e^{.8t^{1/2} - .10t}\left(.4t^{-\frac{1}{2}} - .10\right) = 0$$

$$\frac{.4}{\sqrt{t}} = .10$$

$$\frac{.10\sqrt{t}}{.10} = \frac{.4}{.10}$$

$$\sqrt{t} = 4$$

$$t = 16$$

**EXAMPLE 6 Newton's Law of Cooling**

Let $y$ represent the temperature (in °F) of an object in a room whose temperature is kept at a constant 60°. If the object cools from 100° to 90° in 10 minutes, how much longer will it take for its temperature to decrease to 80°?

**Solution** From Newton's Law of Cooling, you know that the rate of change in $y$ is proportional to the difference between $y$ and 60. This can be written as

$$y' = k(y - 60), \quad 80 \le y \le 100.$$

To solve this differential equation, use separation of variables, as follows.

$$\frac{dy}{dt} = k(y - 60) \qquad \text{Differential equation}$$

$$\left(\frac{1}{y - 60}\right) dy = k\,dt \qquad \text{Separate variables.}$$

$$\int \frac{1}{y - 60}\,dy = \int k\,dt \qquad \text{Integrate each side.}$$

$$\ln|y - 60| = kt + C_1 \qquad \text{Find antiderivative of each side.}$$

Because $y > 60$, $|y - 60| = y - 60$, and you can omit the absolute value signs. Using exponential notation, you have

$$y - 60 = e^{kt + C_1} \quad \Rightarrow \quad y = 60 + Ce^{kt}. \qquad C = e^{C_1}$$

Using $y = 100$ when $t = 0$, you obtain $100 = 60 + Ce^{k(0)} = 60 + C$, which implies that $C = 40$. Because $y = 90$ when $t = 10$,

$$90 = 60 + 40e^{k(10)}$$
$$30 = 40e^{10k}$$
$$k = \tfrac{1}{10}\ln\tfrac{3}{4} \approx -0.02877.$$

So, the model is

$$y = 60 + 40e^{-0.02877t} \qquad \text{Cooling model}$$

and finally, when $y = 80$, you obtain

$$80 = 60 + 40e^{-0.02877t}$$
$$20 = 40e^{-0.02877t}$$
$$\tfrac{1}{2} = e^{-0.02877t}$$
$$\ln\tfrac{1}{2} = -0.02877t$$
$$t \approx 24.09 \text{ minutes.}$$

So, it will require about 14.09 *more* minutes for the object to cool to a temperature of 80° (see Figure 6.11).

73.

***Newton's Law of Cooling*** When an object is removed from a furnace and placed in an environment with a constant temperature of 80°F, its core temperature is 1500°F. One hour after it is removed, the core temperature is 1120°F. Find the core temperature 5 hours after the object is removed from the furnace.

$$y = 80 + Ce^{kt}$$
$$1500 = 80 + Ce^{k(0)}$$
$$1420 = C$$
$$y = 80 + 1420e^{kt}$$
$$1120 = 80 + 1420e^{k(1)}$$
$$1040 = 1420e^{k}$$
$$\frac{104}{142} = e^{k}$$
$$\ln\left(\frac{104}{142}\right) = k$$
$$k \approx -.311436$$