9.2 Series and Convergence

One very useful application of calculus involves the approximation of a function by a sequence of polynomials called **Taylor polynomials**. Before diving into the topic of Taylor polynomials, it will be wise to develop some of the general theory of **series and their convergence**. (9.2-9.6)

Calculators use the Taylor Series or the Cordic algorithm

for sine and rest of 6 trigonometric functions.

```
#include <iostream>
using namespace std;
// exp example
#include <cstdio>      // printf
#include <cmath>       //  exp

double toRadians(double angdeg)         //convert to radians to degree
{                                       //x is in radians
    const double PI = 3.14159265358979323846;
    return angdeg / 180.0 * PI;
}

double fact(double x)       //factorial function
{                           //Simply calculates factorial for denominator
    if(x==0 || x==1)
        return 1;
    else
        x * fact(x - 1);
}

double mySin(double x)      //mySin function
{
    double sum = 0.0;
    for(int i = 0; i < 9; i++)
    {
        double top = pow(-1, i) * pow(x, 2 * i + 1);  //calculation for nominator
        double bottom = fact(2 * i + 1);              //calculation for denominator
        sum = sum + top / bottom;                     //1 - x^2/2! + x^4/4! - x^6/6!
    }
    return sum;
}

int main()
{
```

taylor polynomial
for sine

9.2 Series and Convergence

- Define series
- nth term test for divergence
- ***Geometric series*******
- Telescopiong series-does not come up that often

infinite series (series):

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

adding up infinitely many terms

when we add them up
will the sum converge or diverge?

Converge or Diverge?

$1 + 2 + 3 + 4 + 5 + \ldots \longrightarrow \infty$

$1 + .1 + .01 + .001 + .0001 + \ldots \longrightarrow 1.\overline{1}$

if the things you are adding up are
getting small enough fast enough
then will converge

1
.1
.01
.001
.0001

In general it is a hard question to answer what it converges to, usually just answer whether it converges or diverges?

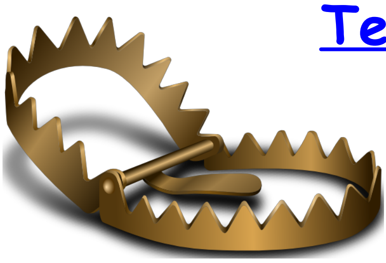
just like integration...if the problem changes a little bit, it can do very different things

Test for Divergence
(nth term test for Divergence)

1st thing you ask yourself

nth term test for divergence

if $\lim_{n\to\infty} a_n \neq 0$ Diverges then you're done



Test for Divergence

The trap:

the opposite is NOT true

$$\lim_{n\to\infty} a_n = 0$$

may converge or diverge, have to investigate further

Test for Divergence

1/2 diverges

Ex. 1

$$\sum_{n=1}^{\infty} \frac{n}{2n+3}$$

$\lim_{n\to\infty} \frac{n}{2n+3}$

$\frac{1}{2} \neq 0$

Diverges by nth term test

Ex. 2

$$\sum_{n=1}^{\infty} \frac{n!}{3n!+2}$$

$$\lim_{n\to\infty} \frac{n!}{3n!+2} = \frac{1}{3} \neq 0$$

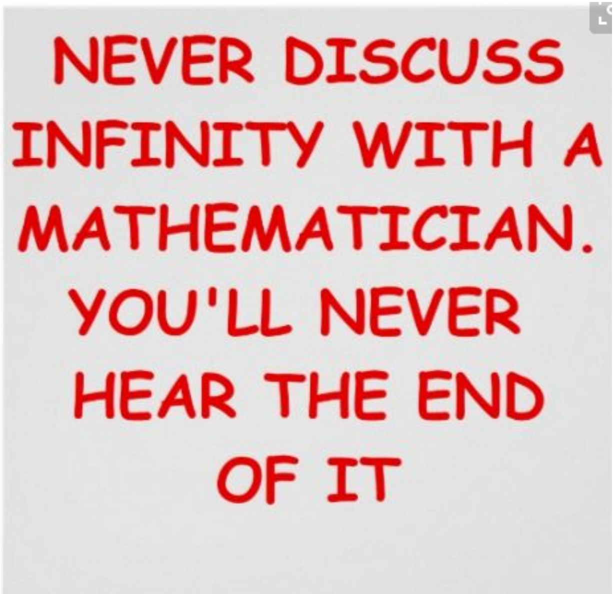
diverges
by nth term test

Ex. 3

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

inconclusive
need another test



Only two types can you actually tell what the series converges to. Both are in this section:

- **Geometric Series*****(see most often)
- Telescoping Series

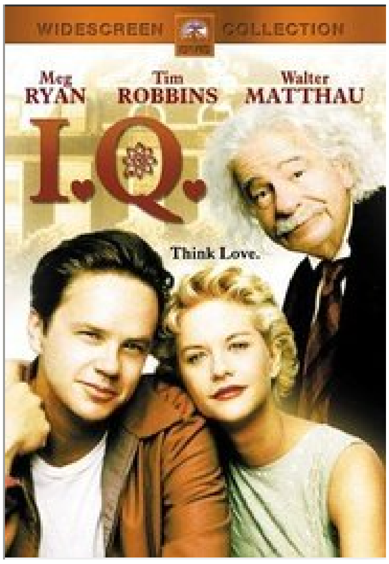
Geometric Series

Objective: Determine whether a geometric series converges or diverges.

If it converges, find the sum the of the series.

http://cleanvideosearch.com/media/action/yt/watch?videoId=EfqVnj-sgcc

(4:11)



Albert Einstein helps a young man who's in love with Einstein's niece to catch her attention by pretending temporarily to be a great physicist. (1994 PG)

http://cleanvideosearch.com/media/action/yt/watch?videoId=YuftF3ZnMZM
play :59

Geometric series numbers raised to powers

1 + .1 + .01 + .001 + .0001 + . . . $=1.\overline{1}$

Could rewrite...

$$1+1\left(\frac{1}{10}\right)+1\left(\frac{1}{10}\right)^2+1\left(\frac{1}{10}\right)^3+1\left(\frac{1}{10}\right)^4+1\left(\frac{1}{10}\right)^5+\ldots$$

when you **multiply** by the same number to get the next term

$n=1$ $n=2$ $n=3$ $n=4$

$$1+1\left(\frac{1}{10}\right)+1\left(\frac{1}{10}\right)^2+1\left(\frac{1}{10}\right)^3+1\left(\frac{1}{10}\right)^4+1\left(\frac{1}{10}\right)^5+\ldots$$

$1\left(\frac{1}{10}\right)^{1-1}$

1

$$\sum_{n=1}^{\infty}1\left(\frac{1}{10}\right)^{n-1}$$

notice if plug in 1, etc.

$$\sum_{n=1}^{\infty}1\left(\frac{1}{10}\right)^{n-1}$$

common ratio = r

<u>converge</u>: $|r|<1$ or $-1<r<1$

otherwise diverges: could be ∞, $-\infty$ or

oscillates like (-1, 1, -1, 1,)

If you have a **geometric series** that converges...

you can tell what it converges to
(in other words you can find
the sum of the series)

To Find the Sum of a Geometric Series

$$\frac{a}{1-r}$$

a = **1st term** (be careful...it is **not** always 1)

$$\frac{1st\ term}{1-common\ ratio}$$

show with previous ex.
put in calculator you'll see 1.1 repeating

$$\frac{1}{1-\frac{1}{10}}$$

$$\frac{1}{\frac{10}{10}-\frac{1}{10}} = \frac{1}{\frac{9}{10}} = \frac{10}{9}$$

NORMAL FLOAT AUTO REAL DEGREE MP

10/9

1.111111111

Ex. 1 **Geometric Series**

Converge or Diverge?

If it converges, find the sum.

$1 + .4 + .16 + .064 + \ldots$

$.4 \quad .4$

$$\frac{1}{1-.4} = \frac{1}{.6}$$

$$\frac{1}{\frac{6}{10}} = \frac{10}{6} = 5/3$$

Ex. 2 Using Geometric Series

to write a rational number

(repeating decimal expansion)

as a fraction.

$5.12121\overline{212}\ldots$

more room next slide

$5.12121\overline{12}...$

$5 + .12 + \underbrace{.0012}_{.01} + \underbrace{.000012}_{.01} + ...$

$5 + \frac{.12}{1-.01}$

Ex. 3 **Geometric Series**

Converge or Diverge?

If it converges, find the sum.

$$\sum_{n=3}^{\infty} 5\left(\frac{2}{3}\right)^{n-1}$$

$\frac{2}{3} < 1$ conv. by geo. series

$$\frac{a}{1-r} = \frac{5\left(\frac{2}{3}\right)^{3-1}}{1-\frac{2}{3}}$$

$$\frac{5\left(\frac{4}{9}\right)}{\frac{1}{3}} = \frac{\frac{20}{9}}{\frac{1}{3}}$$

$$\frac{20}{9} \cdot \frac{3}{1} = 20/3$$

Ex. 4 **Geometric Series**

Converge or Diverge?

If it converges, find the sum.

$$\sum_{n=1}^{\infty} \frac{\pi^n}{3^{n+2}}$$

$$\sum_{n=1}^{\infty} \frac{\pi^n}{3^n \cdot 3^2}$$

$$\sum_{n=1}^{\infty} \left(\frac{\pi}{3}\right)^n \cdot \frac{1}{9}$$

$\frac{\pi}{3} > 1$

diverges by geo. series

Ex. 5 **Geometric Series**

Converge or Diverge?

If it converges, find the sum.

$$\sum_{n=1}^{\infty} \frac{1}{4^n} = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n$$

$\frac{1}{4} < 1$

conv. by geo series

$$\frac{a}{1-r} = \frac{\frac{1}{4}}{1-\frac{1}{4}}$$

$$\frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

comes up A LOT!!

$$\frac{1}{4^n} = \left(\frac{1}{4}\right)^n$$

Rewrite

Ex. 6

$$\sum_{n=1}^{\infty} \frac{3^n + 2^n}{6^n}$$

$$\sum_{n=1}^{\infty} \frac{3^n}{6^n} + \sum_{n=1}^{\infty} \frac{2^n}{6^n}$$

$$\sum_{n=1}^{\infty} \left(\frac{3}{6}\right)^n + \sum_{n=1}^{\infty} \left(\frac{2}{6}\right)^n$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$$

Both conv. by geo. $\frac{1}{2} < 1$ $\frac{1}{3} < 1$

$$\frac{\frac{1}{2}}{1 - \frac{1}{2}} + \frac{\frac{1}{3}}{1 - \frac{1}{3}}$$

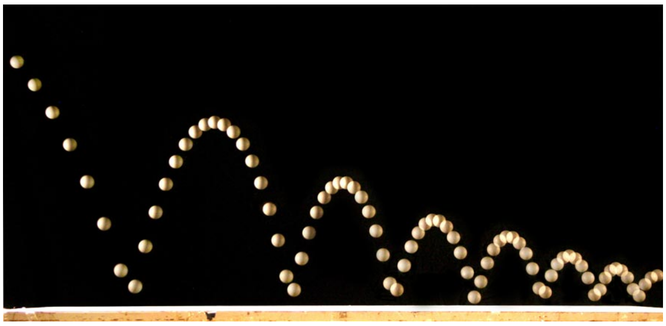
$$\frac{\frac{1}{2}}{\frac{1}{2}} + \frac{\frac{1}{3}}{\frac{2}{3}}$$

$$1 + \frac{1}{2}$$

$$\frac{3}{2}$$

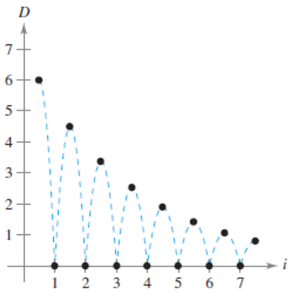
Ex. 7

A ball is dropped from a height of 6 feet and begins bouncing. The height of each bounce is three-fourths the height of the previous bounce. Find the total vertical distance traveled by the ball.



read t

now t



The height of each bounce is three-fourths the height of the preceding bounce.
Figure 9.7

EXAMPLE 6 Bouncing Ball Problem

A ball is dropped from a height of 6 feet and begins bouncing, as shown in Figure 9.7. The height of each bounce is three-fourths the height of the previous bounce. Find the total vertical distance traveled by the ball.

Solution When the ball hits the ground for the first time, it has traveled a distance of $D_1 = 6$ feet. For subsequent bounces, let D_i be the distance traveled up and down. For example, D_2 and D_3 are as follows.

$$D_2 = \underbrace{6\left(\tfrac{3}{4}\right)}_{\text{Up}} + \underbrace{6\left(\tfrac{3}{4}\right)}_{\text{Down}} = 12\left(\tfrac{3}{4}\right)$$

$$D_3 = \underbrace{6\left(\tfrac{3}{4}\right)\left(\tfrac{3}{4}\right)}_{\text{Up}} + \underbrace{6\left(\tfrac{3}{4}\right)\left(\tfrac{3}{4}\right)}_{\text{Down}} = 12\left(\tfrac{3}{4}\right)^2$$

By continuing this process, it can be determined that the total vertical distance is

$$\begin{aligned} D &= 6 + 12\left(\tfrac{3}{4}\right) + 12\left(\tfrac{3}{4}\right)^2 + 12\left(\tfrac{3}{4}\right)^3 + \cdots \\ &= 6 + 12\sum_{n=0}^{\infty}\left(\tfrac{3}{4}\right)^{n+1} \\ &= 6 + 12\left(\tfrac{3}{4}\right)\sum_{n=0}^{\infty}\left(\tfrac{3}{4}\right)^{n} \\ &= 6 + 9\left(\frac{1}{1-\frac{3}{4}}\right) \\ &= 6 + 9(4) \\ &= 42 \text{ feet.} \end{aligned}$$