8.8 Improper Integrals

read p. 580

- What makes an integral improper?
- Classify #5-10 on p. 585

Improper Integrals with Infinite Limits of Integration

The definition of a definite integral

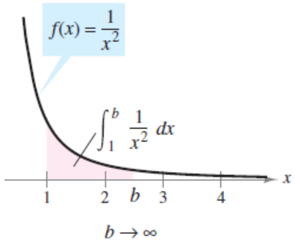
$$\int_a^b f(x)\,dx$$

requires that the interval $[a, b]$ be finite. Furthermore, the Fundamental Theorem of Calculus, by which you have been evaluating definite integrals, requires that f be continuous on $[a, b]$. In this section you will study a procedure for evaluating integrals that do not satisfy these requirements—usually because either one or both of the limits of integration are infinite, or f has a finite number of infinite discontinuities in the interval $[a, b]$. Integrals that possess either property are **improper integrals.** Note that a function f is said to have an **infinite discontinuity** at c if, *from the right or left,*

$$\lim_{x\to c} f(x) = \infty \qquad \text{or} \qquad \lim_{x\to c} f(x) = -\infty.$$

To get an idea of how to evaluate an improper integral, consider the integral

$$\int_1^b \frac{dx}{x^2} = -\frac{1}{x}\Big]_1^b = -\frac{1}{b} + 1 = 1 - \frac{1}{b}$$



unbounded region has an area of 1.
ıre 8.17

which can be interpreted as the area of the shaded region shown in Figure 8.17. Taking the limit as $b \to \infty$ produces

$$\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty}\left(\int_1^b \frac{dx}{x^2}\right) = \lim_{b\to\infty}\left(1 - \frac{1}{b}\right) = 1.$$

This improper integral can be interpreted as the area of the *unbounded* region between the graph of $f(x) = 1/x^2$ and the x-axis (to the right of $x = 1$).

DEFINITION OF IMPROPER INTEGRALS WITH INFINITE INTEGRATION LIMITS

1. If f is continuous on the interval $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

2. If f is continuous on the interval $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

3. If f is continuous on the interval $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

where c is any real number (see Exercise 120).

In the first two cases, the improper integral **converges** if the limit exists—otherwise, the improper integral **diverges.** In the third case, the improper integral on the left diverges if either of the improper integrals on the right diverges.

Ex. 1

$$\int_1^\infty \frac{5}{x^3}\,dx$$

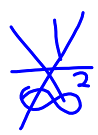
$$\lim_{b\to\infty} \int_1^b 5x^{-3}\,dx$$

$$\lim_{b\to\infty} \left[\frac{5x^{-2}}{-2} \right]_1^b$$

$$-\frac{5}{2}\left[\lim_{b\to\infty} \frac{1}{x^2}\right]_1^b$$

$F(b) - F(1)$

$$-\frac{5}{2}\lim_{b\to\infty}\left[\frac{1}{b^2} - \frac{1}{1^2}\right]$$



$$-\frac{5}{2}\left[0-1\right]$$

$5/2$ converges

Ex. 2

$$\int_9^\infty \frac{2}{\sqrt{x}}\,dx$$

$$\lim_{b\to\infty}\int_9^b 2x^{-1/2}\,dx$$

$$\lim_{b\to\infty}\left[2\cdot 2\cdot x^{1/2}\right]_9^b$$

$$4\lim_{b\to\infty}\left[x^{1/2}\right]_9^b$$

$$F(b)-F(9)$$

$$4\lim_{b\to\infty} b^{1/2}-9^{1/2}$$

$$\downarrow$$

$$\infty$$

Diverges

Ex. 3

$$\int_0^\infty (x-1)e^{-x}\,dx$$

L
I
A $x-1$
T
E $e^{-x}dx$

$u=\underline{x-1}$ $\quad v=-e^{-x}$

$du=dx$ $\quad dv=\underline{e^{-x}dx}$

$$\lim_{b\to\infty}\int_0^b (x-1)e^{-x}\,dx$$

$uv-\int v\,du$

$$\lim_{b\to\infty}\left[(x-1)(-e^{-x})-\int -e^{-x}\,dx\right]_0^b$$

$$\lim_{b\to\infty}\left[-e^{-x}(x-1)+\int e^{-x}\,dx\right]_0^b$$

$$\lim_{b\to\infty}\left[-e^{-x}(x-1)-e^{-x}\right]_0^b$$

$$\lim_{b\to\infty}\left[-xe^{-x}+e^{-x}-e^{-x}\right]_0^b$$

$$\lim_{b\to\infty}\left[-xe^{-x}\right]_0^b$$

$$\lim_{b\to\infty}\left[\frac{-x}{e^x}\right]_0^b$$

L'Hop $\frac{-1}{e^x}$

$$F(b)-F(0)$$

$$\lim_{b\to\infty}\frac{-b}{e^b}-0$$

conv.

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