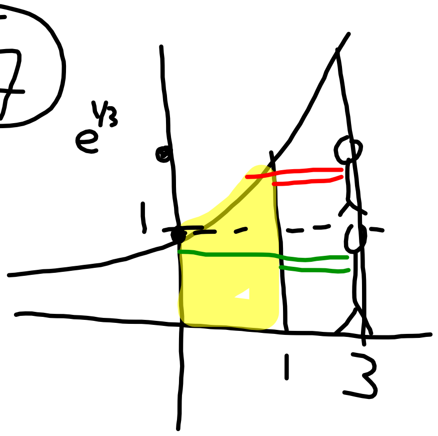
Geometry with Calculus

go over 7 and 8

⑦



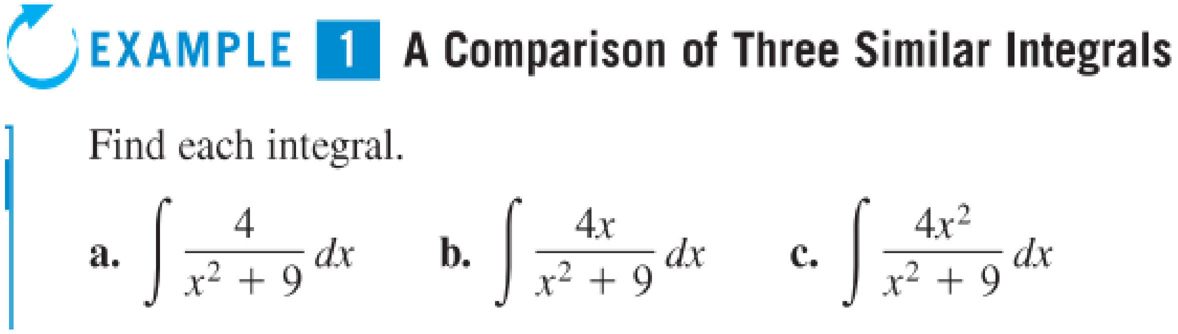
$y = e^{x/3}$

$\ln y = \frac{x}{3}$

$3\ln y = x$

8.1 Basic Integration Rules

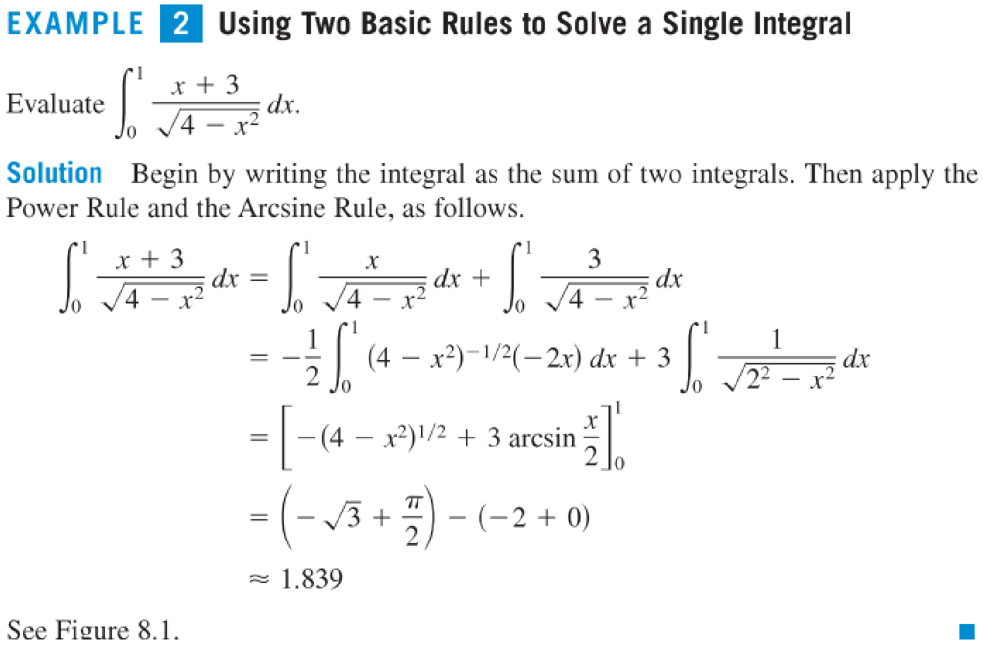
Study examples (1 - 6) on pgs. 520-523
and gray box on p. 522



EXAMPLE 1 A Comparison of Three Similar Integrals

Find each integral.

a. $\displaystyle\int \frac{4}{x^2+9}\,dx$ **b.** $\displaystyle\int \frac{4x}{x^2+9}\,dx$ **c.** $\displaystyle\int \frac{4x^2}{x^2+9}\,dx$



EXAMPLE 2 Using Two Basic Rules to Solve a Single Integral

Evaluate $\displaystyle\int_0^1 \frac{x+3}{\sqrt{4-x^2}}\,dx$.

Solution Begin by writing the integral as the sum of two integrals. Then apply the Power Rule and the Arcsine Rule, as follows.

$$\begin{aligned}\int_0^1 \frac{x+3}{\sqrt{4-x^2}}\,dx &= \int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx + \int_0^1 \frac{3}{\sqrt{4-x^2}}\,dx\\ &= -\frac{1}{2}\int_0^1 (4-x^2)^{-1/2}(-2x)\,dx + 3\int_0^1 \frac{1}{\sqrt{2^2-x^2}}\,dx\\ &= \left[-(4-x^2)^{1/2} + 3\arcsin\frac{x}{2}\right]_0^1\\ &= \left(-\sqrt{3}+\frac{\pi}{2}\right) - (-2+0)\\ &\approx 1.839\end{aligned}$$

See Figure 8.1. ■

EXAMPLE 3 A Substitution Involving $a^2 - u^2$

Find $\displaystyle\int \frac{x^2}{\sqrt{16 - x^6}}\,dx$.

Solution Because the radical in the denominator can be written in the form

$$\sqrt{a^2 - u^2} = \sqrt{4^2 - (x^3)^2}$$

you can try the substitution $u = x^3$. Then $du = 3x^2\,dx$, and you have

$$\int \frac{x^2}{\sqrt{16 - x^6}}\,dx = \frac{1}{3}\int \frac{3x^2\,dx}{\sqrt{16 - (x^3)^2}} \qquad \text{Rewrite integral.}$$

$$= \frac{1}{3}\int \frac{du}{\sqrt{4^2 - u^2}} \qquad \text{Substitution: } u = x^3$$

$$= \frac{1}{3}\arcsin\frac{u}{4} + C \qquad \text{Arcsine Rule}$$

$$= \frac{1}{3}\arcsin\frac{x^3}{4} + C. \qquad \text{Rewrite as a function of } x.$$

■

EXAMPLE 4 A Disguised Form of the Log Rule

Find $\displaystyle\int \frac{1}{1 + e^x}\,dx$.

Solution The integral does not appear to fit any of the basic rules. However, the quotient form suggests the Log Rule. If you let $u = 1 + e^x$, then $du = e^x\,dx$. You can obtain the required du by adding and subtracting e^x in the numerator, as follows.

$$\int \frac{1}{1 + e^x}\,dx = \int \frac{1 + e^x - e^x}{1 + e^x}\,dx \qquad \text{Add and subtract } e^x \text{ in numerator.}$$

$$= \int \left(\frac{1 + e^x}{1 + e^x} - \frac{e^x}{1 + e^x}\right)dx \qquad \text{Rewrite as two fractions.}$$

$$= \int dx - \int \frac{e^x\,dx}{1 + e^x} \qquad \text{Rewrite as two integrals.}$$

$$= x - \ln(1 + e^x) + C \qquad \text{Integrate.}$$

■

NOTE There is usually more than one way to solve an integration problem. For instance, in Example 4, try integrating by multiplying the numerator and denominator by e^{-x} to obtain an integral of the form $-\int du/u$. See if you can get the same answer by this procedure. (Be careful: the answer will appear in a different form.) ■

EXAMPLE 5 A Disguised Form of the Power Rule

Find $\int (\cot x)[\ln(\sin x)]\, dx$.

Solution Again, the integral does not appear to fit any of the basic rules. However, considering the two primary choices for u [$u = \cot x$ and $u = \ln(\sin x)$], you can see that the second choice is the appropriate one because

$$u = \ln(\sin x) \quad \text{and} \quad du = \frac{\cos x}{\sin x}\, dx = \cot x\, dx.$$

So,

$$\begin{aligned} \int (\cot x)[\ln(\sin x)]\, dx &= \int u\, du && \text{Substitution: } u = \ln(\sin x) \\ &= \frac{u^2}{2} + C && \text{Integrate.} \\ &= \frac{1}{2}[\ln(\sin x)]^2 + C. && \text{Rewrite as a function of } x. \end{aligned}$$

■

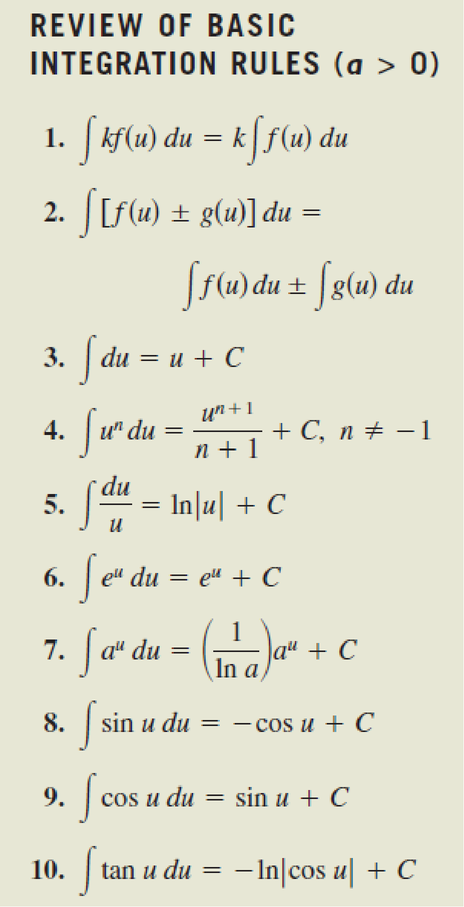
EXAMPLE 6 Using Trigonometric Identities

Find $\int \tan^2 2x\, dx$.

Solution Note that $\tan^2 u$ is not in the list of basic integration rules. However, $\sec^2 u$ is in the list. This suggests the trigonometric identity $\tan^2 u = \sec^2 u - 1$. If you let $u = 2x$, then $du = 2\, dx$ and

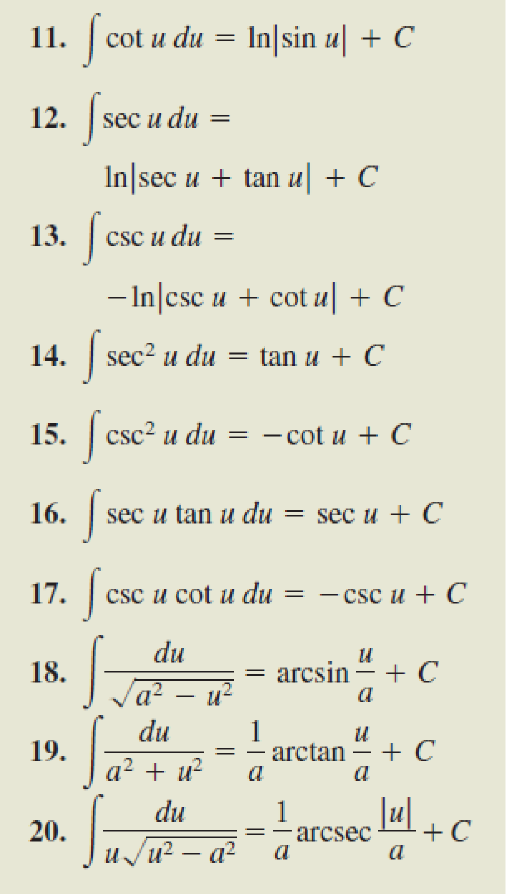
$$\begin{aligned} \int \tan^2 2x\, dx &= \frac{1}{2}\int \tan^2 u\, du && \text{Substitution: } u = 2x \\ &= \frac{1}{2}\int (\sec^2 u - 1)\, du && \text{Trigonometric identity} \\ &= \frac{1}{2}\int \sec^2 u\, du - \frac{1}{2}\int du && \text{Rewrite as two integrals.} \\ &= \frac{1}{2}\tan u - \frac{u}{2} + C && \text{Integrate.} \\ &= \frac{1}{2}\tan 2x - x + C. && \text{Rewrite as a function of } x. \end{aligned}$$

■

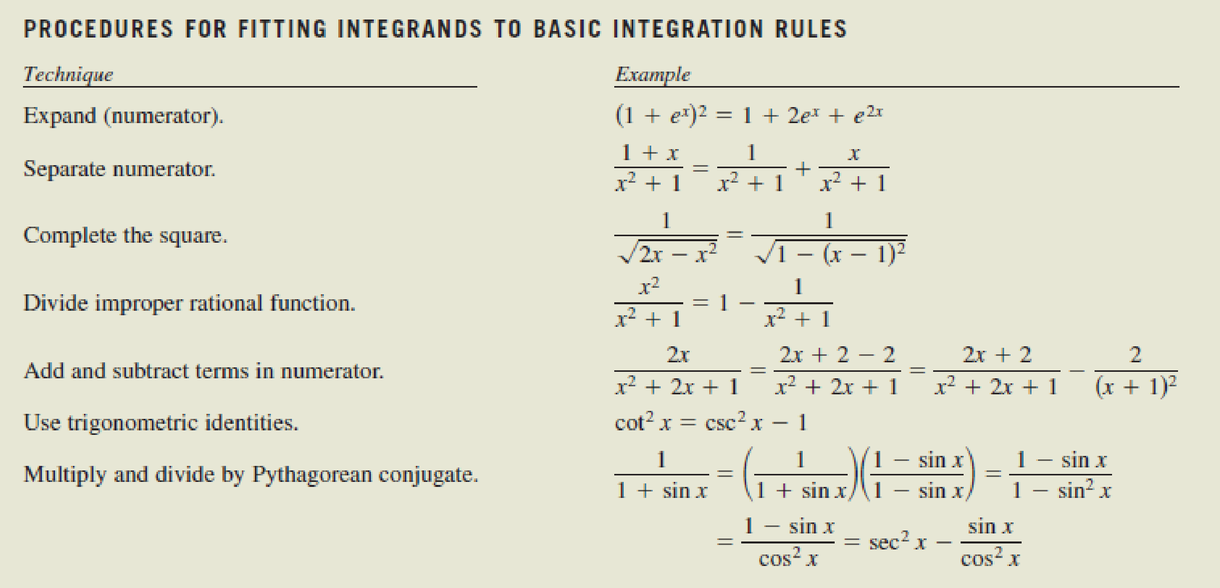


REVIEW OF BASIC INTEGRATION RULES ($a > 0$)

1. $\int kf(u)\,du = k\int f(u)\,du$
2. $\int [f(u) \pm g(u)]\,du = \int f(u)\,du \pm \int g(u)\,du$
3. $\int du = u + C$
4. $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C,\ n \neq -1$
5. $\int \dfrac{du}{u} = \ln|u| + C$
6. $\int e^u\,du = e^u + C$
7. $\int a^u\,du = \left(\dfrac{1}{\ln a}\right)a^u + C$
8. $\int \sin u\,du = -\cos u + C$
9. $\int \cos u\,du = \sin u + C$
10. $\int \tan u\,du = -\ln|\cos u| + C$

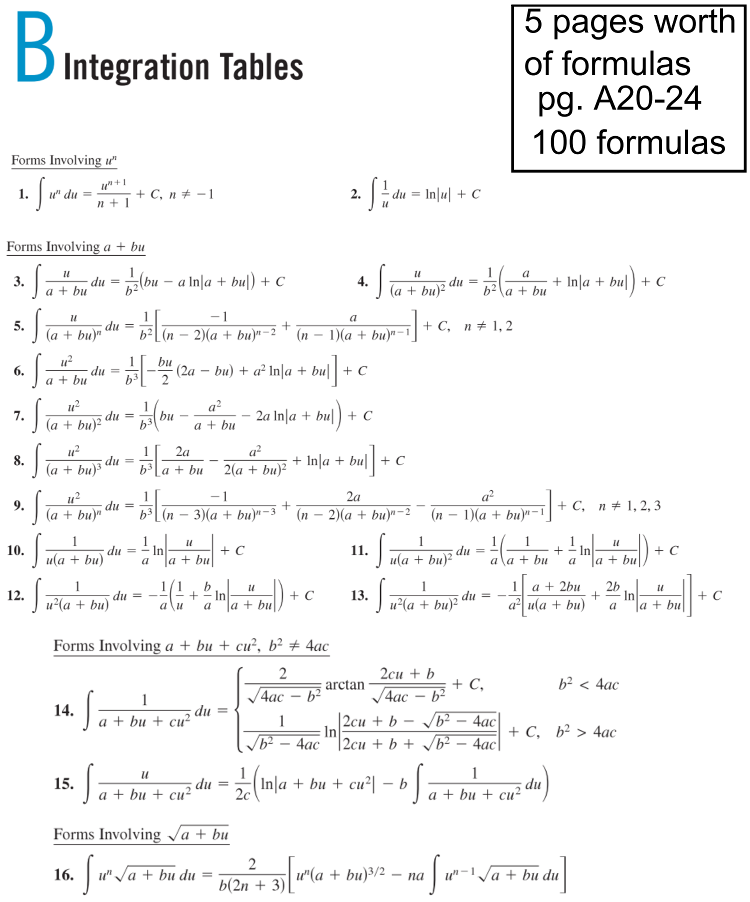


11. $\int \cot u\,du = \ln|\sin u| + C$
12. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
13. $\int \csc u\,du = -\ln|\csc u + \cot u| + C$
14. $\int \sec^2 u\,du = \tan u + C$
15. $\int \csc^2 u\,du = -\cot u + C$
16. $\int \sec u \tan u\,du = \sec u + C$
17. $\int \csc u \cot u\,du = -\csc u + C$
18. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \arcsin \dfrac{u}{a} + C$
19. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a}\arctan \dfrac{u}{a} + C$
20. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a}\operatorname{arcsec}\dfrac{|u|}{a} + C$



PROCEDURES FOR FITTING INTEGRANDS TO BASIC INTEGRATION RULES

Technique	*Example*
Expand (numerator).	$(1 + e^x)^2 = 1 + 2e^x + e^{2x}$
Separate numerator.	$\dfrac{1 + x}{x^2 + 1} = \dfrac{1}{x^2 + 1} + \dfrac{x}{x^2 + 1}$
Complete the square.	$\dfrac{1}{\sqrt{2x - x^2}} = \dfrac{1}{\sqrt{1 - (x - 1)^2}}$
Divide improper rational function.	$\dfrac{x^2}{x^2 + 1} = 1 - \dfrac{1}{x^2 + 1}$
Add and subtract terms in numerator.	$\dfrac{2x}{x^2 + 2x + 1} = \dfrac{2x + 2 - 2}{x^2 + 2x + 1} = \dfrac{2x + 2}{x^2 + 2x + 1} - \dfrac{2}{(x + 1)^2}$
Use trigonometric identities.	$\cot^2 x = \csc^2 x - 1$
Multiply and divide by Pythagorean conjugate.	$\dfrac{1}{1 + \sin x} = \left(\dfrac{1}{1 + \sin x}\right)\left(\dfrac{1 - \sin x}{1 - \sin x}\right) = \dfrac{1 - \sin x}{1 - \sin^2 x}$ $= \dfrac{1 - \sin x}{\cos^2 x} = \sec^2 x - \dfrac{\sin x}{\cos^2 x}$



B Integration Tables

5 pages worth
of formulas
pg. A20-24
100 formulas

Forms Involving u^n

1. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C,\ n \neq -1$

2. $\int \frac{1}{u}\,du = \ln|u| + C$

Forms Involving $a + bu$

3. $\int \frac{u}{a+bu}\,du = \frac{1}{b^2}\left(bu - a\ln|a+bu|\right) + C$

4. $\int \frac{u}{(a+bu)^2}\,du = \frac{1}{b^2}\left(\frac{a}{a+bu} + \ln|a+bu|\right) + C$

5. $\int \frac{u}{(a+bu)^n}\,du = \frac{1}{b^2}\left[\frac{-1}{(n-2)(a+bu)^{n-2}} + \frac{a}{(n-1)(a+bu)^{n-1}}\right] + C,\quad n \neq 1, 2$

6. $\int \frac{u^2}{a+bu}\,du = \frac{1}{b^3}\left[-\frac{bu}{2}(2a - bu) + a^2\ln|a+bu|\right] + C$

7. $\int \frac{u^2}{(a+bu)^2}\,du = \frac{1}{b^3}\left(bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C$

8. $\int \frac{u^2}{(a+bu)^3}\,du = \frac{1}{b^3}\left[\frac{2a}{a+bu} - \frac{a^2}{2(a+bu)^2} + \ln|a+bu|\right] + C$

9. $\int \frac{u^2}{(a+bu)^n}\,du = \frac{1}{b^3}\left[\frac{-1}{(n-3)(a+bu)^{n-3}} + \frac{2a}{(n-2)(a+bu)^{n-2}} - \frac{a^2}{(n-1)(a+bu)^{n-1}}\right] + C,\quad n \neq 1, 2, 3$

10. $\int \frac{1}{u(a+bu)}\,du = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$

11. $\int \frac{1}{u(a+bu)^2}\,du = \frac{1}{a}\left(\frac{1}{a+bu} + \frac{1}{a}\ln\left|\frac{u}{a+bu}\right|\right) + C$

12. $\int \frac{1}{u^2(a+bu)}\,du = -\frac{1}{a}\left(\frac{1}{u} + \frac{b}{a}\ln\left|\frac{u}{a+bu}\right|\right) + C$

13. $\int \frac{1}{u^2(a+bu)^2}\,du = -\frac{1}{a^2}\left[\frac{a+2bu}{u(a+bu)} + \frac{2b}{a}\ln\left|\frac{u}{a+bu}\right|\right] + C$

Forms Involving $a + bu + cu^2,\ b^2 \neq 4ac$

14. $\int \frac{1}{a+bu+cu^2}\,du = \begin{cases} \frac{2}{\sqrt{4ac-b^2}}\arctan\frac{2cu+b}{\sqrt{4ac-b^2}} + C, & b^2 < 4ac \\ \frac{1}{\sqrt{b^2-4ac}}\ln\left|\frac{2cu+b-\sqrt{b^2-4ac}}{2cu+b+\sqrt{b^2-4ac}}\right| + C, & b^2 > 4ac \end{cases}$

15. $\int \frac{u}{a+bu+cu^2}\,du = \frac{1}{2c}\left(\ln|a+bu+cu^2| - b\int \frac{1}{a+bu+cu^2}\,du\right)$

Forms Involving $\sqrt{a + bu}$

16. $\int u^n\sqrt{a+bu}\,du = \frac{2}{b(2n+3)}\left[u^n(a+bu)^{3/2} - na\int u^{n-1}\sqrt{a+bu}\,du\right]$

Find the indefinite integrals.

$\int t^2\sqrt[3]{t^3-1}dt$	$\int \frac{2x}{x-4}dx$
$\int (\tan x)[\ln(\cos x)]dx$	$\int \csc^2 x \cdot e^{\cot x}dx$
$\int \frac{1}{x\sqrt{x^2-4}}dx$	$\int \frac{x^2}{x-1}dx$

① $\int t^2 \sqrt[3]{t^3-1}\,dt$

$u = t^3-1$

$du = 3t^2\,dt$

$\frac{1}{3}du = t^2\,dt$

$\frac{1}{3}\int u^{1/3}\,du$

$\cancel{\frac{1}{3}}\cdot\frac{\cancel{3}}{4}\cdot u^{4/3}$

$\frac{1}{4}(t^3-1)^{4/3} + C$

② $\int \frac{2x}{x-4}\,dx$

$u = x-4$ $x = u+4$

$2\int \frac{x}{x-4}\,dx$

$2\int \frac{u+4}{u}\,du$

$2\int 1 + \frac{4}{u}\,du$

$2\left[u + 4\ln|u|\right] + C$

$\boxed{2(x-4) + 8\ln|x-4| + C}$

or long $2\int \frac{x}{x-4}\,dx$

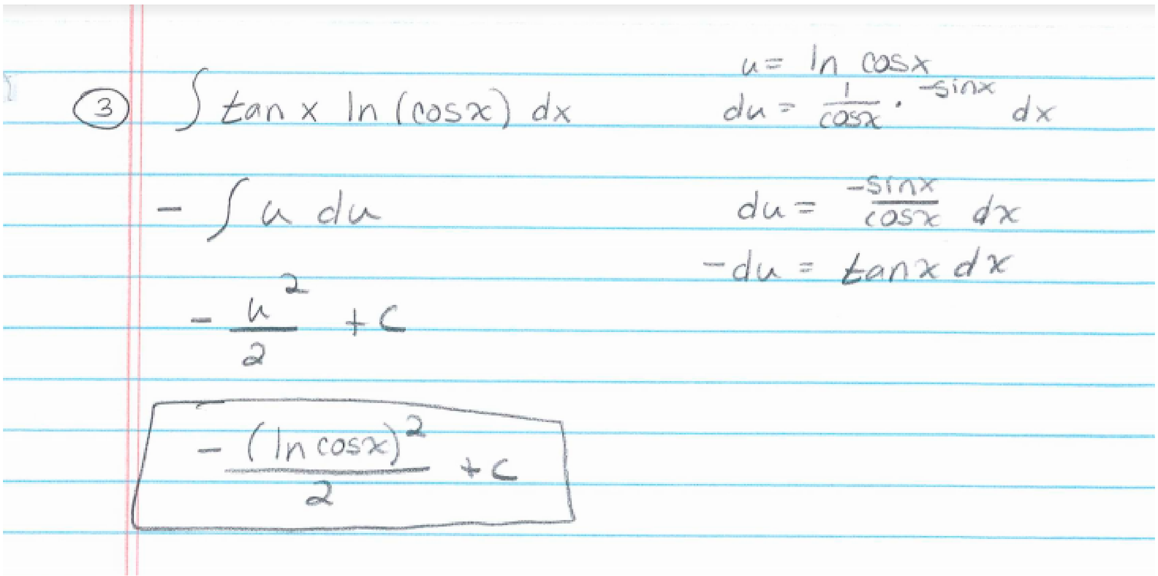
$$\begin{array}{r|l} & 1 + \frac{4}{x-4} \\ \hline x-4 & x \\ & -x \pm 4 \\ \hline & 4 \end{array}$$

or long division

$$\begin{array}{r|l} & 2 + \frac{8}{x-4} \\ \hline x-4 & 2x \\ & -2x + 8 \\ \hline \end{array}$$

$\int 2 + \frac{8}{x-4}\,dx$

$\boxed{2x + 8\ln|x-4| + C}$



③ $\int \tan x \ln(\cos x)\, dx$

$u = \ln \cos x$

$du = \frac{1}{\cos x} \cdot -\sin x \, dx$

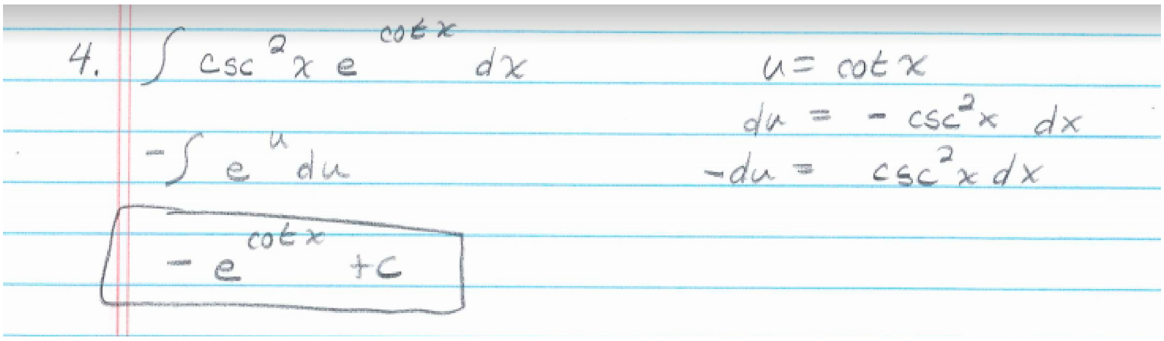
$du = \frac{-\sin x}{\cos x}\, dx$

$-du = \tan x\, dx$

$-\int u\, du$

$-\frac{u^2}{2} + c$

$\boxed{-\frac{(\ln \cos x)^2}{2} + c}$



4. $\int \csc^2 x\, e^{\cot x}\, dx$

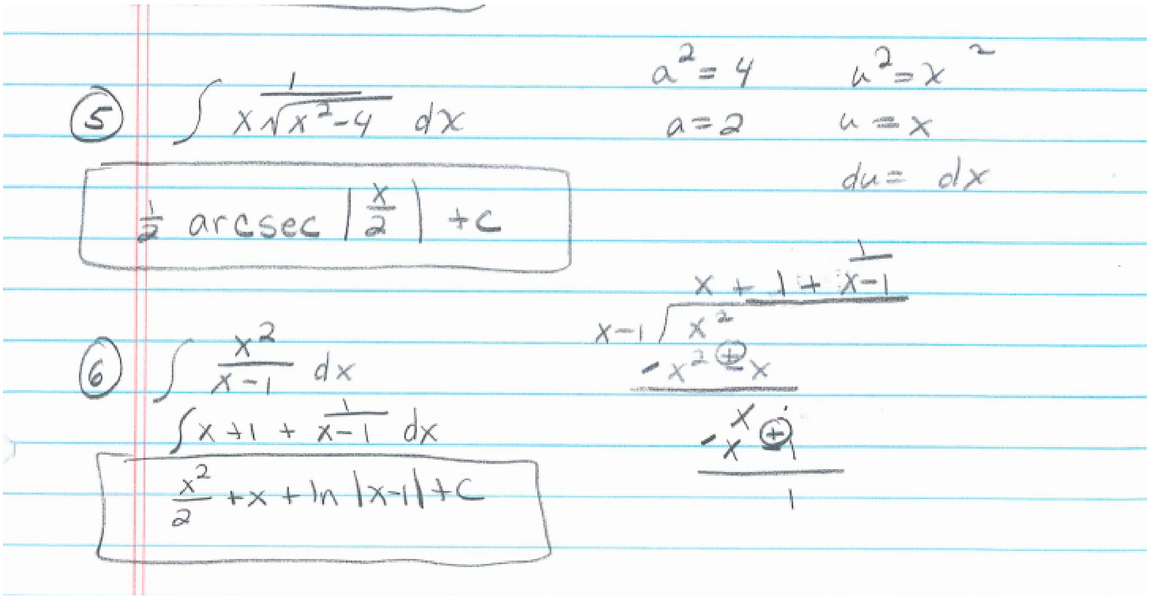
$u = \cot x$

$du = -\csc^2 x\, dx$

$-du = \csc^2 x\, dx$

$-\int e^u\, du$

$\boxed{-e^{\cot x} + c}$



⑤ $\int \frac{1}{x\sqrt{x^2-4}}\,dx$

$a^2 = 4$

$a = 2$

$u^2 = x^2$

$u = x$

$du = dx$

$$\boxed{\frac{1}{2}\,\text{arcsec}\left|\frac{x}{2}\right| + C}$$

⑥ $\int \frac{x^2}{x-1}\,dx$

$$\begin{array}{r|l} & x + 1 + \frac{1}{x-1} \\ x-1 & x^2 \\ & -x^2 \oplus x \\ & x \\ & -x \oplus 1 \\ & 1 \end{array}$$

$\int x + 1 + \frac{1}{x-1}\,dx$

$$\boxed{\frac{x^2}{2} + x + \ln|x-1| + C}$$

Make a "game plan" for integration!

Turn in for a grade!!!!!!!